

JUMP-TYPE HUNT PROCESSES GENERATED BY LOWER BOUNDED SEMI-DIRICHLET FORMS

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Let E be a locally compact separable metric space and m be a positive Radon measure on it. Given a nonnegative function k defined on $E \times E$ off the diagonal whose anti-symmetric part is assumed to be less singular than the symmetric part, we construct an associated regular lower bounded semi-Dirichlet form η on $L^2(E; m)$ producing a Hunt process X^0 on E whose jump behaviours are governed by k . For an arbitrary open subset $D \subset E$, we also construct a Hunt process $X^{D,0}$ on D in an analogous manner. When D is relatively compact, we show that $X^{D,0}$ is censored in the sense that it admits no killing inside D and killed only when the path approaches to the boundary. When E is a d -dimensional Euclidean space and m is the Lebesgue measure, a typical example of X^0 is the stable-like process that will be also identified with the solution of a martingale problem up to an η -polar set of starting points. Approachability to the boundary ∂D in finite time of its censored process $X^{D,0}$ on a bounded open subset D will be examined in terms of the polarity of ∂D for the symmetric stable processes with indices that bound the variable exponent $\alpha(x)$.

1. Introduction. Let E be a locally compact separable metric space equipped with a metric d , m be a positive Radon measure with full topological support and $k(x, y)$ be a nonnegative Borel measurable function on the space $E \times E \setminus \text{diag}$, where diag denotes the diagonal set $\{(x, x) : x \in E\}$. A purpose of the present paper is to construct Hunt processes on E and on its subsets with jump behaviors being governed by the kernel k by using general results on a lower bounded semi-Dirichlet form on $L^2(E; m)$.

The inner product and the norm in $L^2(E; m)$ are denoted by (\cdot, \cdot) and $\|\cdot\|$, respectively. Let \mathcal{F} be a dense linear subspace of $L^2(E; m)$ such that $u \wedge 1 \in \mathcal{F}$ whenever $u \in \mathcal{F}$. A (not necessarily symmetric) bilinear form η on \mathcal{F} is called

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a *lower bounded closed form* if the following three conditions are satisfied: we set $\eta_\beta(u, v) = \eta(u, v) + \beta(u, v)$, $u, v \in \mathcal{F}$. There exists a $\beta_0 \geq 0$ such that:

- (B.1) (lower boundedness); for any $u \in \mathcal{F}$, $\eta_{\beta_0}(u, u) \geq 0$.
- (B.2) (sector condition); for any $u, v \in \mathcal{F}$,

$$|\eta(u, v)| \leq K \sqrt{\eta_{\beta_0}(u, u)} \cdot \sqrt{\eta_{\beta_0}(v, v)}$$

for some constant $K \geq 1$.

(B.3) (completeness); the space \mathcal{F} is complete with respect to the norm $\eta_\alpha^{1/2}(\cdot, \cdot)$ for some, or equivalently, for all $\alpha > \beta_0$.

For a lower bounded closed form (η, \mathcal{F}) on $L^2(E; m)$, there exist unique semigroups $\{T_t; t > 0\}, \{\widehat{T}_t; t > 0\}$ of linear operators on $L^2(E; m)$ satisfying

$$(1.1) \quad \begin{aligned} (T_t f, g) &= (f, \widehat{T}_t g), \\ f, g &\in L^2(E; m), \|T_t\| \leq e^{\beta_0 t}, \|\widehat{T}_t\| \leq e^{\beta_0 t}, t > 0, \end{aligned}$$

such that their Laplace transforms G_α and \widehat{G}_α are determined for $\alpha > \beta_0$ by

$$\begin{aligned} G_\alpha f, \widehat{G}_\alpha f &\in \mathcal{F}, \quad \eta_\alpha(G_\alpha f, u) = \eta_\alpha(u, \widehat{G}_\alpha f) = (f, u), \\ f &\in L^2(E; m), u \in \mathcal{F}. \end{aligned}$$

See the first part of Section 3 for more details. $\{T_t; t > 0\}$ is said to be *Markovian* if $0 \leq T_t f \leq 1, t > 0$, whenever $f \in L^2(E; m), 0 \leq f \leq 1$. It was shown by Kunita [15] that the semigroup $\{T_t; t > 0\}$ is Markovian if and only if

$$(1.2) \quad Uu \in \mathcal{F} \quad \text{and} \quad \eta(Uu, u - Uu) \geq 0 \quad \text{for any } u \in \mathcal{F},$$

where Uu denotes the unit contraction of u : $Uu = (0 \vee u) \wedge 1$. A lower bounded closed form (η, \mathcal{F}) on $L^2(E; m)$ satisfying (1.2) will be called a *lower bounded semi-Dirichlet form* on $L^2(E; m)$. The term “semi” is added to indicate that the dual semigroup $\{\widehat{T}_t; t > 0\}$ may not be Markovian although it is positivity preserving. As we shall see in Section 3 for a lower bounded semi-Dirichlet form η which is regular in the sense stated below, if the associated dual semigroup $\{\widehat{T}_t; t > 0\}$ were Markovian, or equivalently, if m were excessive, then η is necessarily a nonnegative definite closed form, namely, β_0 in conditions (B.1), (B.3) [resp., (B.2)] can be retaken to be 0 (resp., 1).

A lower bounded semi-Dirichlet form (η, \mathcal{F}) is said to be *regular* if $\mathcal{F} \cap C_0(E)$ is uniformly dense in $C_0(E)$ and η_α -dense in \mathcal{F} for $\alpha > \beta_0$, where $C_0(E)$ denotes the space of continuous functions on E with compact support. Carrillo-Menendez [8] constructed a Hunt process properly associated with any regular lower bounded semi-Dirichlet form on $L^2(E; m)$ by reducing the situation to the case where η is nonnegative definite. We shall show in Section 4 that a direct construction is possible without such a reduction.

Later on, the nonnegative definite semi-Dirichlet form was investigated by Ma, Oberbeck and Röckner [16] and Fitzsimmons [10] specifically in a general context of the quasi-regular Dirichlet form and the special standard process. However, in producing the forms η from nonsymmetric kernels k corresponding to a considerably wide class of jump type Hunt processes in finite dimensions whose dual semigroups need not be Markovian, we will be forced to allow positive β_0 .

To be more precise, we set for $x, y \in E, x \neq y$,

$$(1.3) \quad k_s(x, y) := \frac{1}{2}\{k(x, y) + k(y, x)\} \quad \text{and} \quad k_a(x, y) := \frac{1}{2}\{k(x, y) - k(y, x)\},$$

that is, the kernel $k_s(x, y)$ denotes the symmetrized one of k , while $k_a(x, y)$ represents the anti-symmetric part of k . We impose four conditions (2.1)–(2.4) on k_s and k_a stated below. Condition (2.1) on k_s is nearly optimal for us to work with the symmetric Dirichlet form (1.4) defined below, while conditions (2.2)–(2.4) require k_a to be less singular than k_s .

Let conditions (2.1)–(2.4) be in force on k . Denote by $C_0^{\text{lip}}(E)$ the space of uniformly Lipschitz continuous functions on E with compact support. We also let

$$(1.4) \quad \begin{cases} \mathcal{E}(u, v) := \int \int_{E \times E \setminus \text{diag}} (u(y) - u(x))(v(y) - v(x)) \\ \quad \times k_s(x, y)m(dx)m(dy), \\ \mathcal{F}^r = \{u \in L^2(E; m) : u \text{ is Borel measurable and } \mathcal{E}(u, u) < \infty\}. \end{cases}$$

$(\mathcal{E}, \mathcal{F}^r)$ is a symmetric Dirichlet form on $L^2(E; m)$ and \mathcal{F}^r contains the space $C_0^{\text{lip}}(E)$. We denote by \mathcal{F}^0 the \mathcal{E}_1 -closure of $C_0^{\text{lip}}(E)$ in \mathcal{F}^r . $(\mathcal{E}, \mathcal{F}^0)$ is then a regular Dirichlet form on $L^2(E; m)$ (cf. [13], Example 1.2.4, Theorem 3.1.1 and see also [23] and [24]).

For $u \in C_0^{\text{lip}}(E)$ and $n \in \mathbb{N}$, the integral

$$(1.5) \quad \mathcal{L}^n u(x) := \int_{\{y \in E : d(x, y) > 1/n\}} (u(y) - u(x))k(x, y)m(dy), \quad x \in E,$$

makes sense. We prove in Proposition 2.1 and Theorem 2.1 in Section 2 that the finite limit

$$(1.6) \quad \eta(u, v) = - \lim_{n \rightarrow \infty} \int_E \mathcal{L}^n u(x)v(x)m(dx) \quad \text{for } u, v \in C_0^{\text{lip}}(E),$$

exists, η extends to $\mathcal{F}^0 \times \mathcal{F}^0$ and (η, \mathcal{F}^0) is a lower bounded semi-Dirichlet form on $L^2(E; m)$ with parameter $\beta_0 = 8(C_1 \vee C_2 C_3)(\geq 0)$ where C_1 – C_3 are constants appearing in conditions (2.2)–(2.4). Furthermore, the form \mathcal{E} is shown to be a *reference (symmetric Dirichlet) form* of η in the sense that, for each fixed $\alpha > \beta_0$,

$$(1.7) \quad c_1 \mathcal{E}_1(u, u) \leq \eta_\alpha(u, u) \leq c_2 \mathcal{E}_1(u, u), \quad u \in \mathcal{F}^0,$$

for some positive constants c_1, c_2 independent of $u \in \mathcal{F}^0$. Therefore, (η, \mathcal{F}^0) becomes a regular lower bounded semi-Dirichlet form on $L^2(E; m)$ and gives

rise to an associated Hunt process $X^0 = (X_t^0, P_x^0)$ on E . We call X^0 the *minimal Hunt process* associated with the form η . Equation (1.6) indicates that the limit of \mathcal{L}^n in n plays a role of a pre-generator of X^0 informally.

If we define the kernel k^* by

$$(1.8) \quad k^*(x, y) := k(y, x), \quad x, y \in E, x \neq y,$$

and the form η^* by (1.5) and (1.6) with k^* in place of k , we have the same conclusions as above for η^* (Corollary 2.1 of Section 2). In particular, there exists a minimal Hunt process X^{0*} associated with the form η^* .

In the second half of Section 3, we are concerned with a killed dual semigroup $\{e^{-\beta t} \widehat{T}_t; t > 0\}$, which can be verified to be Markovian for a large $\beta > 0$ but only for a restricted subfamily of the forms η considered in Section 2 (lower order cases). For a higher order η , the killed dual semigroup may not be Markovian no matter how big β is. We shall also exhibit an example of a one-dimensional probability kernel k [$\int_{\mathbb{R}^1} k(x, y) dy = 1$] with m being the Lebesgue measure, for which the associated semi-Dirichlet form η is not nonnegative definite and accordingly the associated dual semigroup itself is non-Markovian.

When $E = \mathbb{R}^d$ the d -dimensional Euclidean space and $m(dx) = dx$ the Lebesgue measure on it, we shall verify in Section 5 that our requirements (2.1)–(2.4) on the kernel $k(x, y)$ are fulfilled by

$$(1.9) \quad \begin{aligned} k^{(1)}(x, y) &= w(x)|x - y|^{-d-\alpha(x)}, \\ k^{(1)*}(x, y) &= w(y)|x - y|^{-d-\alpha(y)}, \quad x, y \in \mathbb{R}^d, x \neq y, \end{aligned}$$

for $w(x)$ given by (5.1) and $\alpha(x)$ satisfying the bounds (5.2). A Markov process corresponding to $k^{(1)}$ is called a *stable-like process* and has been constructed by Bass [4] as a unique solution to a martingale problem. In this case, we shall prove that the minimal Hunt process associated with the corresponding form η is conservative and actually a solution to the same martingale problem, identifying it with the one constructed in [4] up to an η -polar set of starting points.

In Section 6, we consider an arbitrary open subset D of E . Define m_D by $m_D(B) = m(B \cap D)$ for any Borel set $B \subset E$. By replacing E and m with D and m_D , respectively, in (1.4), we obtain a symmetric Dirichlet form $(\mathcal{E}_D, \mathcal{F}_D^r)$ on $L^2(D; m_D)$. Denote by \overline{D} the closure of D and by $C_0^{\text{lip}}(\overline{D})$ the restriction to \overline{D} of the space $C_0^{\text{lip}}(E)$. We also denote by $C_0^{\text{lip}}(D)$ the space of uniformly Lipschitz continuous functions on D with compact support in D . Let $\mathcal{F}_{\overline{D}}$ and \mathcal{F}_D^0 be the $\mathcal{E}_{D,1}$ -closures of $C_0^{\text{lip}}(\overline{D})$ and $C_0^{\text{lip}}(D)$, respectively, in \mathcal{F}_D^r . Then $(\mathcal{E}_D, \mathcal{F}_{\overline{D}})$ is a regular symmetric Dirichlet form on $L^2(\overline{D}; m_D)$, while $(\mathcal{E}_D^0, \mathcal{F}_D^0)$ is a regular symmetric Dirichlet form on $L^2(D; m_D)$ where \mathcal{E}_D^0 is the restriction of \mathcal{E}_D to $\mathcal{F}_D^0 \times \mathcal{F}_D^0$.

By making the same replacement in (1.5) and (1.6), we get a form η_D on $C_0^{\text{lip}}(\overline{D}) \times C_0^{\text{lip}}(\overline{D})$, which extends to $\mathcal{F}_{\overline{D}} \times \mathcal{F}_{\overline{D}}$ to be a regular lower

bounded semi-Dirichlet form on $L^2(\overline{D}; m_D)$ possessing \mathcal{E}_D as its reference form, yielding an associated Hunt process $X^{\overline{D}}$ on \overline{D} . We also consider the restriction η_D^0 of η_D to $\mathcal{F}_D^0 \times \mathcal{F}_D^0$ so that $(\eta_D^0, \mathcal{F}_D^0)$ is a regular lower bounded semi-Dirichlet form on $L^2(D; m_D)$ possessing \mathcal{E}_D^0 as its reference form. We shall show in Section 6 that the part process $X^{D,0}$ of $X^{\overline{D}}$ on D , namely, the Hunt process obtained from $X^{\overline{D}}$ by killing upon hitting the boundary ∂D , is properly associated with $(\eta_D^0, \mathcal{F}_D^0)$.

We shall also prove in Section 6 that $X^{\overline{D}}$ admits no jump from D to ∂D , and furthermore when D is relatively compact, $X^{\overline{D}}$ is conservative so that $X^{D,0}$ admits no killing inside D and its sample path is killed only when it approaches to the boundary ∂D . $X^{D,0}$ is accordingly different from the part process of X^0 on the set D in general because the sample path of X^0 may jump from D to $E \setminus D$ resulting in a killing inside D of its part process. By adopting k^* instead of k , we get in an analogous manner Hunt processes $X^{\overline{D}*}$ on \overline{D} and $X^{D,0*}$ on D satisfying the same properties as above.

When $(\mathcal{E}, \mathcal{F}^r)$ is the Dirichlet form on $L^2(\mathbb{R}^d)$ of a symmetric stable process on \mathbb{R}^d , the space \mathcal{F}^0 is identical with \mathcal{F}^r . In this case, for an arbitrary open set $D \subset \mathbb{R}^d$, the symmetric Hunt process on D associated with $(\mathcal{E}_D^0, \mathcal{F}_D^0)$ is a *censored stable process* on D in the sense of Bogdan, Burdzy and Chen [7]. It was further shown in [7] that, if D is a d -set, then the space \mathcal{F}_D^0 coincides with \mathcal{F}_D^r so that the symmetric Hunt process on \overline{D} associated with $(\mathcal{E}_D, \mathcal{F}_D^r)$ was called a *reflecting stable process* over \overline{D} .

For the nonsymmetric kernel $k^{(1)}$ on \mathbb{R}^d as (1.9), associated Hunt processes $X^{D,0}, X^{D,0*}$ on an arbitrary open set $D \subset \mathbb{R}^d$ may well be called *censored stable-like processes* in view of the stated properties of them. However, it is harder in this case to identify the space \mathcal{F}_D^0 with \mathcal{F}_D^r , and accordingly we call the associated Hunt processes $X^{\overline{D}}, X^{\overline{D}*}$ over \overline{D} *modified reflecting stable-like processes* analogously to the Brownian motion case (cf. [11]). At the end of Section 6, we give sufficient conditions in terms of the upper and lower bounds of the variable exponent $\alpha(x)$ for the approachability in finite time of the censored stable-like processes to the boundary.

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2. Construction of a lower bounded semi-Dirichlet form from k . Throughout this section, we make the following assumptions on a nonnegative Borel measurable function $k(x, y)$ on $E \times E \setminus \text{diag}$:

$$(2.1) \quad M_s \in L_{\text{loc}}^2(E; m) \quad \text{for } M_s(x) = \int_{y \neq x} (1 \wedge d(x, y)^2) k_s(x, y) m(dy),$$

$$x \in E,$$

$$(2.2) \quad C_1 := \sup_{x \in E} \int_{d(x,y) \geq 1} |k_a(x,y)| m(dy) < \infty,$$

and there exists a constant $\gamma \in (0, 1]$ such that

$$(2.3) \quad C_2 := \sup_{x \in E} \int_{d(x,y) < 1} |k_a(x,y)|^\gamma m(dy) < \infty,$$

and furthermore, for some constant $C_3 \geq 0$,

$$(2.4) \quad |k_a(x,y)|^{2-\gamma} \leq C_3 k_s(x,y) \quad \text{for any } x, y \in E$$

with $0 < d(x,y) \leq 1$.

For each $n \in \mathbb{N}$, define $\mathcal{L}^n u$ for $u \in C_0^{\text{lip}}(E)$ by (1.5) and $\eta^n(u, v)$ for $u, v \in C_0^{\text{lip}}(E)$ by

$$(2.5) \quad \eta^n(u, v) := - \int_E \mathcal{L}^n u(x) v(x) m(dx),$$

the integral on the right-hand side being absolutely convergent by (2.1). We note that any $u \in C_0^{\text{lip}}(E)$ belongs to the domain \mathcal{F}^r of the form \mathcal{E} defined by (1.4). In fact, if we denote by K the support of u , then $\mathcal{E}(u, u)$ is dominated by twice the integral of $(u(x) - u(y))^2 k_s(x, y) m(dx) m(dy)$ on $K \times E$, which is finite by (2.1).

$\mathcal{E}(u, v)$ admits also an alternative expression for $u, v \in C_0^{\text{lip}}(E)$,

$$\mathcal{E}(u, v) := \iint_{E \times E \setminus \text{diag}} (u(y) - u(x))(v(y) - v(x)) k(x, y) m(dx) m(dy),$$

because the right-hand side of the above can be seen to be equal to the same integral with $k(y, x)$ in place of $k(x, y)$ by interchanging the variables x, y , and we arrive at the expression in (1.4) by averaging. In particular, $\mathcal{E}(u, v) = \lim_{n \rightarrow \infty} \mathcal{E}^n(u, v)$ for $u, v \in C_0^{\text{lip}}(E)$ where

$$(2.6) \quad \mathcal{E}^n(u, v) := \iint_{d(x,y) > 1/n} (u(y) - u(x))(v(y) - v(x)) k(x, y) m(dx) m(dy).$$

PROPOSITION 2.1. *Assume (2.1)–(2.4). Then for all $u, v \in C_0^{\text{lip}}(E)$, the limit*

$$\eta(u, v) = \lim_{n \rightarrow \infty} \eta^n(u, v)$$

exists. Moreover, the limit has the following expression:

$$(2.7) \quad \eta(u, v) = \frac{1}{2} \mathcal{E}(u, v) + \iint_{y \neq x} (u(x) - u(y)) v(y) k_a(x, y) m(dx) m(dy),$$

where \mathcal{E} is defined by (1.4) and the integral on the right-hand side is absolutely convergent.

PROOF. For $u, v \in C_0^{\text{lip}}(E)$, we have

$$\begin{aligned}
 \eta^n(u, v) - \eta^n(v, u) &= - \iint_{d(x, y) > 1/n} (u(y) - u(x))v(x)k(x, y)m(dx)m(dy) \\
 &\quad + \iint_{d(x, y) > 1/n} (v(y) - v(x))u(x)k(x, y)m(dx)m(dy) \\
 &= - \iint_{d(x, y) > 1/n} u(y)v(x)k(x, y)m(dx)m(dy) \\
 &\quad + \iint_{d(x, y) > 1/n} v(y)u(x)k(x, y)m(dx)m(dy) \\
 &= 2 \iint_{d(x, y) > 1/n} u(x)v(y)k_a(x, y)m(dx)m(dy),
 \end{aligned}$$

and further

$$\begin{aligned}
 \eta^n(u, v) + \eta^n(v, u) &= - \iint_{d(x, y) \geq 1/n} (u(y) - u(x))v(x)k(x, y)m(dx)m(dy) \\
 &\quad - \iint_{d(x, y) \geq 1/n} (v(y) - v(x))u(x)k(x, y)m(dx)m(dy) \\
 &= \iint_{d(x, y) \geq 1/n} (u(y) - u(x))(v(y) - v(x))k(x, y)m(dx)m(dy) \\
 &\quad - \iint_{d(x, y) \geq 1/n} (u(y) - u(x))v(y)k(x, y)m(dx)m(dy) \\
 &\quad - \iint_{d(x, y) \geq 1/n} (v(y) - v(x))u(x)k(x, y)m(dx)m(dy) \\
 &= \mathcal{E}^n(u, v) - 2 \iint_{d(x, y) \geq 1/n} u(y)v(y)k_a(x, y)m(dx)m(dy).
 \end{aligned}$$

By adding up the obtained identities, we get for $u, v \in C_0^{\text{lip}}(E)$,

$$\begin{aligned}
 2\eta^n(u, v) &= \mathcal{E}^n(u, v) + 2 \iint_{d(x, y) > 1/n} (u(x) - u(y))v(y) \\
 (2.8) \quad &\quad \times k_a(x, y)m(dx)m(dy).
 \end{aligned}$$

Since $\mathcal{E}^n(u, v)$ converges to $\mathcal{E}(u, v)$ as $n \rightarrow \infty$, it remains to see that the second term of the right-hand side also converges absolutely as $n \rightarrow \infty$ for each $u, v \in C_0^{\text{lip}}(E)$.

From the Schwarz inequality and (2.2), we see that

$$\begin{aligned}
& \iint_{d(x,y)>1/n} |(u(x) - u(y))v(y)k_a(x, y)|m(dx)m(dy) \\
& \leq \iint_{1/n < d(x,y) < 1} |u(x) - u(y)| \cdot |v(y)| |k_a(x, y)|^{\gamma/2} \\
& \quad \times |k_a(x, y)|^{1-\gamma/2} m(dx)m(dy) \\
& \quad + \iint_{d(x,y) \geq 1} |u(x) - u(y)| \cdot |v(y)| k_s(x, y)^{1/2} |k_a(x, y)|^{1/2} m(dx)m(dy) \\
& \leq \sqrt{\iint_{1/n < d(x,y) < 1} (u(x) - u(y))^2 |k_a(x, y)|^{2-\gamma} m(dx)m(dy)} \\
& \quad \times \sqrt{\iint_{1/n < d(x,y) < 1} v(y)^2 |k_a(x, y)|^\gamma m(dx)m(dy)} \\
& \quad + \sqrt{C_1} \|v\| \sqrt{\iint_{d(x,y) \geq 1} (u(x) - u(y))^2 k_s(x, y) m(dx)m(dy)}.
\end{aligned}$$

So, by making use of assumptions (2.3) and (2.4) and an elementary inequality $\sqrt{A} + \sqrt{B} \leq \sqrt{2}\sqrt{A+B}$ holding for $A \geq 0$ and $B \geq 0$, we have

$$\begin{aligned}
& \iint_{d(x,y)>1/n} |(u(x) - u(y))v(y)k_a(x, y)|m(dx)m(dy) \\
& \leq \sqrt{2}\sqrt{C_1 \vee C_2 C_3} \|v\| \cdot \sqrt{\mathcal{E}^n(u, u)}.
\end{aligned}$$

Then taking $n \rightarrow \infty$,

$$\begin{aligned}
& \iint_{y \neq x} |(u(x) - u(y))v(y)k_a(x, y)|m(dx)m(dy) \\
& \leq \sqrt{2}\sqrt{C_1 \vee C_2 C_3} \|v\| \cdot \sqrt{\mathcal{E}(u, u)} < \infty
\end{aligned}$$

as was to be proved. \square

For $u, v \in C_0^{\text{lip}}(E)$, set

$$\eta_\beta(u, v) = \eta(u, v) + \beta(u, v), \quad \beta > 0,$$

and

$$(2.9) \quad B(u, v) := \iint_{x \neq y} (u(x) - u(y))v(y)k_a(x, y)m(dx)m(dy).$$

Then equation (2.7) reads

$$(2.10) \quad \eta(u, v) = \frac{1}{2}\mathcal{E}(u, v) + B(u, v), \quad u, v \in C_0^{\text{lip}}(E),$$

while we get from the proof of the preceding proposition

$$(2.11) \quad |B(u, v)| \leq C_4 \|v\| \sqrt{\mathcal{E}(u, u)},$$

where $C_4 = \sqrt{2} \cdot \sqrt{C_1 \vee C_2 C_3}$. Now we put $\beta_0 := 4(C_4)^2 = 8(C_1 \vee C_2 C_3)$.

From equation (2.10) and the bound (2.11), we have for $u \in C_0^{\text{lip}}(E)$,

$$\begin{aligned} \eta_{\beta_0}(u, u) &= \frac{1}{4}\mathcal{E}_{\beta_0}(u, u) + \frac{1}{4}\mathcal{E}(u, u) + \frac{3}{4}\beta_0 \|u\|^2 + B(u, u) \\ &\geq \frac{1}{4}\mathcal{E}_{\beta_0}(u, u) + \sqrt{3}C_4 \sqrt{\mathcal{E}(u, u)} \|u\| + B(u, u) \geq \frac{1}{4}\mathcal{E}_{\beta_0}(u, u). \end{aligned}$$

Further, for $u, v \in C_0^{\text{lip}}(E)$,

$$\begin{aligned} |\eta(u, v)| &\leq \frac{1}{2}|\mathcal{E}(u, v)| + |B(u, v)| \\ &\leq \frac{1}{2}\sqrt{\mathcal{E}(u, u)}\sqrt{\mathcal{E}(v, v)} + C_4 \|v\| \sqrt{\mathcal{E}(u, u)} \\ &\leq \frac{1}{2}(\sqrt{\mathcal{E}(v, v)} + 2C_4 \|v\|) \sqrt{\mathcal{E}(u, u)} \\ &\leq \frac{\sqrt{2}}{2} \sqrt{\mathcal{E}_{\beta_0}(v, v)} \sqrt{\mathcal{E}_{\beta_0}(u, u)}. \end{aligned}$$

So it also follows that

$$(2.12) \quad |\eta(u, v)| \leq 2\sqrt{2} \sqrt{\eta_{\beta_0}(u, u)} \sqrt{\eta_{\beta_0}(v, v)}$$

and

$$(2.13) \quad \frac{1}{4}\mathcal{E}_{\beta_0}(u, u) \leq \eta_{\beta_0}(u, u) \leq \frac{2+\sqrt{2}}{2}\mathcal{E}_{\beta_0}(u, u), \quad u, v \in C_0^{\text{lip}}(E).$$

Let \mathcal{F}^0 be the \mathcal{E}_1 -closure of $C_0^{\text{lip}}(E)$ in \mathcal{F}^r . Since \mathcal{F}^0 is complete with respect to \mathcal{E}_α for any $\alpha > 0$, the estimates obtained above readily lead us to the first conclusion of the following theorem.

THEOREM 2.1. *Assume (2.1)–(2.4). Then the form η defined by Proposition 2.1 extends from $C_0^{\text{lip}}(E) \times C_0^{\text{lip}}(E)$ to $\mathcal{F}^0 \times \mathcal{F}^0$ to be a lower bounded closed form on $L^2(E; m)$ satisfying (B.1)–(B.3) with $\beta_0 = 8(C_1 \vee C_2 C_3)$, $K = 2\sqrt{2}$ and possessing $(\mathcal{E}, \mathcal{F}^0)$ as a reference form in the sense of (1.7).*

Furthermore, the pair (η, \mathcal{F}^0) is a regular lower bounded semi-Dirichlet form on $L^2(E; m)$.

We note that the above constant β_0 is equal to 0 if k is symmetric: $k(x, y) = k(y, x)$, $(x, y) \in E \times E \setminus \text{diag}$.

PROOF OF THEOREM 2.1. It suffices to prove the contraction property (1.2) for the present pair (η, \mathcal{F}^0) . We first show this for $u \in C_0^{\text{lip}}(E)$. Note

that $Uu \in C_0^{\text{lip}}(E)$ and, for $n \in \mathbb{N}$,

$$\begin{aligned}
& \eta^n(Uu, u - Uu) \\
&= - \iint_{d(x,y) > 1/n} (Uu(y) - Uu(x))(u(x) - Uu(x))k(x,y)m(dx)m(dy) \\
&= \iint_{\{d(x,y) > 1/n\} \cap \{x: u(x) \geq 1\}} (1 - Uu(y))(u(x) - 1)k(x,y)m(dx)m(dy) \\
&\quad - \iint_{\{d(x,y) > 1/n\} \cap \{x: u(x) \leq 0\}} Uu(y)u(x)k(x,y)m(dx)m(dy) \\
&\geq 0.
\end{aligned}$$

Then, we have by Proposition 2.1

$$\eta(Uu, u - Uu) = \lim_{n \rightarrow \infty} \eta^n(Uu, u - Uu) \geq 0.$$

Following a method in [17], Lemma 4.9, we next prove (1.2) for any $u \in \mathcal{F}^0$. Choose a sequence $\{u_\ell\} \subset C_0^{\text{lip}}(E)$ which is \mathcal{E}_1 -convergent to u . Then

$$(2.14) \quad \|Uu_\ell - Uu\| \rightarrow 0, \quad \ell \rightarrow \infty,$$

because U is easily seen to be a continuous operator from $L^2(E; m)$ to $L^2(E; m)$. Fix $\alpha > \beta_0$. We then get from (1.7) the boundedness

$$\sup_\ell \eta_\alpha(Uu_\ell, Uu_\ell) \leq C_2 \sup_\ell \mathcal{E}_1(u_\ell, u_\ell) < \infty.$$

On the other hand, using the dual resolvent \widehat{G}_α associated with the lower bounded closed form (η, \mathcal{F}^0) , we see from equation (3.1) below that, for any $g \in L^2(E; m)$,

$$\eta_\alpha(Uu_\ell, \widehat{G}_\alpha g) = (Uu_\ell, g) \rightarrow (Uu, g) = \eta_\alpha(Uu, \widehat{G}_\alpha g), \quad \ell \rightarrow \infty.$$

Since $\{\widehat{G}_\alpha g : g \in L^2(E, m)\}$ is η_α -dense in \mathcal{F}^0 , we can conclude by making use of the above η_α -bound of $\{Uu_\ell\}$ and the sector condition (B.2) that $\{Uu_\ell\}$ is η_α -weakly convergent to Uu as $\ell \rightarrow \infty$. In particular, by the above η_α -bound and (B.2) again, we have

$$(2.15) \quad \eta_\alpha(Uu_\ell, u_\ell) \rightarrow \eta_\alpha(Uu, u), \quad \ell \rightarrow \infty.$$

We consider the dual form $\widehat{\eta}$ and the symmetrizing form $\tilde{\eta}$ of η defined by

$$\widehat{\eta}(u, v) = \eta(v, u), \quad \tilde{\eta}(u, v) = \frac{1}{2}(\eta(u, v) + \eta(v, u)), \quad u, v \in \mathcal{F}^0.$$

In the same way as above, we can see that $\{Uu_\ell\}$ converges as $\ell \rightarrow \infty$ to Uu $\widehat{\eta}_\alpha$ -weakly and consequently $\tilde{\eta}_\alpha$ -weakly. Since $(\tilde{\eta}_\alpha, \mathcal{F}^0)$ is a nonnegative definite symmetric bilinear form, it follows that

$$\begin{aligned}
(2.16) \quad & \eta_\alpha(Uu, Uu) = \tilde{\eta}_\alpha(Uu, Uu) \leq \liminf_{\ell \rightarrow \infty} \tilde{\eta}_\alpha(Uu_\ell, Uu_\ell) \\
& = \liminf_{\ell \rightarrow \infty} \eta_\alpha(Uu_\ell, Uu_\ell).
\end{aligned}$$

We can then obtain (1.2) for $u \in \mathcal{F}^0$ from (2.14), (2.15) and (2.16) as

$$\begin{aligned} \eta(Uu, u - Uu) &\geq \lim_{\ell \rightarrow \infty} \eta(Uu_\ell, u_\ell) - \liminf_{\ell \rightarrow \infty} \eta(Uu_\ell, Uu_\ell) \\ &= \limsup_{\ell \rightarrow \infty} \eta(Uu_\ell, u_\ell - Uu_\ell) \geq 0. \end{aligned} \quad \square$$

For the kernel k^* defined by (1.8), we have obviously

$$(2.17) \quad k_s^*(x, y) = k_s(x, y) \quad \text{and} \quad k_a^*(x, y) = -k_a(x, y), \quad x, y \in E, x \neq y.$$

Hence, if the kernel $k(x, y)$ satisfies (2.1)–(2.4), so does the kernel $k^*(x, y)$. Define η^* as in Proposition 2.1 with $k^*(x, y)$ in place of $k(x, y)$. The same calculations made above for $k(x, y)$ remain valid for $k^*(x, y)$. Note also that the domain \mathcal{F}^{0*} is the same as \mathcal{F}^0 since the symmetric form \mathcal{E}^* defined by k^* is also the same as \mathcal{E} . Thus, we can have the following corollary.

COROLLARY 2.1. *Assume conditions (2.1)–(2.4) hold. Then the pair (η^*, \mathcal{F}^0) is also a regular lower bounded semi-Dirichlet form on $L^2(E; m)$.*

3. Markov property of dual semigroups. First, we fix a general lower bounded closed form (η, \mathcal{F}) on $L^2(E; m)$ satisfying (B.1)–(B.3) and make several remarks on it. The last condition (B.3) is equivalent to

$$(B.3)' \quad (\tilde{\eta}_{\beta_0}, \mathcal{F}) \text{ is a closed symmetric form on } L^2(E; m),$$

where $\tilde{\eta}$ denotes the symmetrization of the form η : $\tilde{\eta}(u, v) = \frac{1}{2}(\eta(u, v) + \eta(v, u))$. η_{β_0} is therefore a coercive closed form in the sense of [17], Definition 2.4, so that, by [17], Theorem 2.8, there exist uniquely two families of linear bounded operators $\{G_\alpha\}_{\alpha > \beta_0}, \{\hat{G}_\alpha\}_{\alpha > \beta_0}$ on $L^2(E; m)$ such that, for $\alpha > \beta_0$, $G_\alpha(L^2(E; m))$, $\hat{G}_\alpha(L^2(E; m)) \subset \mathcal{F}$ and

$$(3.1) \quad \eta_\alpha(G_\alpha f, u) = (f, u) = \eta_\alpha(u, \hat{G}_\alpha f), \quad f \in L^2(E; m), u \in \mathcal{F}.$$

In particular, G_α and \hat{G}_α are mutually adjoint:

$$(3.2) \quad (G_\alpha g, f) = (g, \hat{G}_\alpha f), \quad f, g \in L^2(E; m), \alpha > \beta_0.$$

We call $\{G_\alpha; \alpha > \beta_0\}$ (resp., $\{\hat{G}_\alpha; \alpha > \beta_0\}$) the *resolvent* (resp., *dual resolvent*) associated with (η, \mathcal{F}) .

Accordingly we see in exactly the same way as the proof of Theorem 2.8 of [17] that there exist strongly continuous contraction semigroups $\{S_t; t > 0\}, \{\hat{S}_t; t > 0\}$ of linear operators on $L^2(E; m)$ such that, for $\alpha > 0, f \in L^2(E; m)$,

$$G_{\beta_0 + \alpha} f = \int_0^\infty e^{-\alpha t} S_t f dt, \quad \hat{G}_{\beta_0 + \alpha} f = \int_0^\infty e^{-\alpha t} \hat{S}_t f dt.$$

We then set $T_t = e^{\beta_0 t} S_t$, $\widehat{T}_t = e^{\beta_0 t} \widehat{S}_t$ to get strongly continuous semigroups $\{T_t; t > 0\}$, $\{\widehat{T}_t; t > 0\}$ satisfying

$$(3.3) \quad G_\alpha f = \int_0^\infty e^{-\alpha t} T_t f dt, \quad \widehat{G}_\alpha f = \int_0^\infty e^{-\alpha t} \widehat{T}_t f dt, \quad \alpha > \beta_0,$$

as well as (1.1).

We call $\{T_t; t > 0\}$ (resp., $\{\widehat{T}_t; t > 0\}$) the *semigroup* (resp., dual semigroup) on $L^2(E; m)$ associated with the lower bounded closed form (η, \mathcal{F}) . We introduce the *dual form* $\widehat{\eta}$ of η by

$$\widehat{\eta}(u, v) = \eta(v, u), \quad u, v \in \mathcal{F}.$$

Then $(\widehat{\eta}, \mathcal{F})$ is a lower bounded closed form on $L^2(E; m)$ with which $\{\widehat{T}_t; t > 0\}$ and $\{\widehat{G}_\alpha; \alpha > \beta_0\}$ are the associated semigroup and resolvent, respectively.

Suppose (η, \mathcal{F}) is a lower bounded semi-Dirichlet form, namely, it satisfies the contraction property (1.2) additionally. As in the proof of the corollary to Theorem 4.1 of [15] or the proof of Theorem 4.4 of [17], we can then readily verify that the family $\{\alpha G_\alpha; \alpha > \beta_0\}$ is Markovian, which is in turn equivalent to the Markovian property of $\{T_t; t > 0\}$. Together with $\{T_t; t > 0\}$, its Laplace transform then determines a bounded linear operator G_α on $L^\infty(E; m)$ for every $\alpha > 0$ and $\{\alpha G_\alpha; \alpha > 0\}$ becomes Markovian. Further, $\{\widehat{T}_t; t > 0\}$ is positivity preserving in view of (1.1).

Suppose additionally that (η, \mathcal{F}) is regular. Then the associated Markovian semigroup and resolvent can be represented by the transition function $\{P_t; t > 0\}$ and the resolvent $\{R_\alpha; \alpha > 0\}$ of the associated Hunt process X specified in Theorem 2 of the next section: $P_t f = T_t f$, $t > 0$, and $R_\alpha f = G_\alpha f$, $\alpha > 0$, for any $f \in \mathcal{B}_b(E) \cap L^2(E; m)$. We call a σ -finite measure μ on E *excessive* relative to X if $\mu P_t \leq \mu$ for any $t > 0$. The next lemma was already observed in Silverstein [20].

LEMMA 3.1. *Let η be a regular lower bounded semi-Dirichlet form on $L^2(E; m)$.*

- (i) *The following three conditions are mutually equivalent:*
 1. *m is excessive relative to X .*
 2. *The dual semigroup $\{\widehat{T}_t; t > 0\}$ is Markovian.*
 3. *$\eta(u - Uu, Uu) \geq 0$ for any $u \in \mathcal{F}$.*
- (ii) *If one of the three conditions in (i) is satisfied, then η is nonnegative definite and the constant β_0 in conditions (B.1), (B.3) [resp., (B.2)] can be retaken to be 0 (resp., 1).*

PROOF. (i) 3 is the Markovian criterion (1.2) for the dual semigroup. If 2 is satisfied, then for any $f \in L^2(E; m)$ with $0 \leq f \leq 1$, $0 \leq \widehat{T}_t f \leq 1$ so that $(f, P_t h) = (\widehat{T}_t f, h) \leq (1, h)$ for any $h \in \mathcal{B}_+ \cap L^2(E; m)$, from which 1 follows. The converse can be shown similarly.

(ii) By the Schwarz inequality,

$$(R_\alpha f(x))^2 \leq R_\alpha 1(x) R_\alpha f^2(x) \leq \frac{1}{\alpha} R_\alpha f^2(x), \quad x \in E, f \in \mathcal{B}_b(E) \cap L^2(E; m).$$

Assuming 1 of (i), an integration with respect to m yields $\alpha^2 \|G_\alpha f\|^2 \leq \|f\|^2$, the L^2 -contraction property of αG_α . In view of [17], Theorem 2.13, $\eta(u, u) = \lim_{\alpha \rightarrow \infty} \alpha(u - \alpha G_\alpha u, u)u \in \mathcal{F}$, which particularly implies that $\eta(u, u) \geq 0$, $u \in \mathcal{F}$, and $\{\eta_\alpha; \alpha > 0\}$ become equivalent on \mathcal{F} . \square

We now return to the setting of the preceding section that (η, \mathcal{F}^0) is defined in terms of the kernel k satisfying conditions (2.1)–(2.4). By Proposition 2.1, $\hat{\eta}(u, v) = \frac{1}{2}\mathcal{E}(v, u) + B(v, u)$ where B is defined by (2.9) on $\mathcal{F}^0 \times \mathcal{F}^0$. On the other hand, we have from (2.17) that $\eta^*(u, v) = \frac{1}{2}\mathcal{E}(u, v) - B(u, v)$ and consequently

$$(3.4) \quad \hat{\eta}(u, v) = \eta^*(u, v) + (B(u, v) + B(v, u)), \quad u, v \in \mathcal{F}^0.$$

We know from Theorem 2.1 and Corollary 2.1 that both (η, \mathcal{F}^0) and (η^*, \mathcal{F}^0) are regular lower bounded semi-Dirichlet forms. In order to get a similar property for the dual form $\hat{\eta}$, we need to impose on the kernel k stronger conditions than (2.1)–(2.4) making the additional term on the right-hand side of (3.4) controllable.

In the rest of this section, we assume that the kernel k satisfies the condition

$$(3.5) \quad M_s \in L_{\text{loc}}^2(E; m) \quad \text{for } M_s(x) = \int_{y \neq x} (1 \wedge d(x, y)) k_s(x, y) m(dy), \quad x \in E,$$

in place of (2.1), and further satisfies condition (2.2) as well as (2.3) for $\gamma = 1$ so that

$$(3.6) \quad \begin{aligned} \frac{\beta_1}{2} &:= \sup_{x \in E} \int_{x \neq y} |k_a(x, y)| m(dy) \\ &= \sup_{x \in E} \frac{1}{2} \int_{x \neq y} |k(x, y) - k(y, x)| m(dy) < \infty. \end{aligned}$$

Notice that condition (2.4) for $\gamma = 1$ is always satisfied with $C_3 = 1$.

Then the integrals

$$(3.7) \quad \begin{aligned} \mathcal{L}u(x) &= \int_{y \neq x} (u(y) - u(x)) k(x, y) m(dy) \quad \text{and} \\ \mathcal{L}^*u(x) &= \int_{y \neq x} (u(y) - u(x)) k^*(x, y) m(dy), \end{aligned}$$

converge for $u \in C_0^{\text{lip}}(E)$, $x \in E$, and we get from Proposition 2.1 the identities

$$(3.8) \quad \eta(u, v) = -(\mathcal{L}u, v), \quad \eta^*(u, v) = -(\mathcal{L}^*u, v), \quad u, v \in C_0^{\text{lip}}(E).$$

Furthermore,

$$(3.9) \quad \begin{aligned} K(x) &:= 2 \int_{y \neq x} k_a(x, y) m(dy) \\ &= \int_{y \neq x} (k(x, y) - k(y, x)) m(dy), \quad x \in E, \end{aligned}$$

defines a bounded function on E and (3.4) readily leads us to

$$\hat{\eta}(u, v) = \eta^*(u, v) + (u, Kv), \quad u, v \in \mathcal{F}^0,$$

which combined with (3.7) means that $\hat{\mathcal{L}} = \mathcal{L}^* - K$ is the formal adjoint of \mathcal{L} . $\hat{\eta}$ does not necessarily satisfy the contraction property (1.2), but the form

$$\hat{\eta}_\beta(u, v) = \eta^*(u, v) + (u, (K + \beta)v), \quad \beta \geq \beta_1,$$

does because so does the form η^* by Corollary 2.1 and $K + \beta \geq 0$ if $\beta \geq \beta_1$. So we have the following proposition.

PROPOSITION 3.1. *Assume that (3.5) and (3.6) hold. Then $(\hat{\eta}_\beta, \mathcal{F}^0)$, which is the dual of $(\eta_\beta, \mathcal{F}^0)$, is a regular lower bounded semi-Dirichlet form on $L^2(E; m)$ provided that $\beta \geq \beta_1$.*

This proposition means that, under conditions (3.5) and (3.6), $\{e^{-\beta t} \hat{T}_t; t > 0\}$ is Markovian for the dual semigroup $\{\hat{T}_t; t > 0\}$ associated with η when $\beta \geq \beta_1$. If (3.6) fails, the dual semigroup of $\{e^{-\beta t} T_t; t > 0\}$ may not be Markovian no matter how large β is.

A nonnegative Borel function k on $E \times E$ is said to be a *probability kernel* if $\int_E k(x, y) m(dy) = 1, x \in E$. A probability kernel k with the additional property

$$(3.10) \quad \sup_{x \in E} \int_D k(y, x) m(dy) < \infty$$

satisfies conditions (3.5) and (3.6) and η defined by (3.8) yields a regular lower bounded semi-Dirichlet form on $L^2(E; m)$. We now give an example of a such a kernel on \mathbb{R}^1 with m being the Lebesgue measure for which the associated semi-Dirichlet form η is *not* nonnegative definite so that, according to Lemma 3.1, the associated dual semigroup $\{\hat{T}_t, t > 0\}$ is *not* Markovian although $\{e^{-\beta t} \hat{T}_t; t > 0\}$ is Markovian for a large $\beta > 0$ in view of

Proposition 3.1. A transition probability density function with respect to the Lebesgue measure of the one-dimensional Brownian motion with a mildly localized drift serves to be an example of such a kernel k .

Consider a diffusion Y on \mathbb{R}^1 with generator $\mathcal{G}u = \frac{1}{2}u'' + \lambda b(x)u'$ where λ is a positive constant and b is a function in $C_0^1(\mathbb{R}^1)$ not identically 0. Then $\mathcal{G} = \frac{d}{dm} \cdot \frac{d}{ds}$ for

$$dm(x) = m(x) dx, \quad ds(x) = 2m(x)^{-1} dx,$$

where

$$m(x) = 2 \exp \left\{ 2\lambda \int_0^x b(y) dy \right\},$$

namely, Y is a diffusion with canonical scale s and canonical (speed) measure dm .

The following facts about Y are taken from [12]. Since $m(x)$ is bounded from above and from below by positive constants, both $\pm\infty$ are nonapproachable in the sense that $s(\pm\infty) = \pm\infty$. Therefore, Y is recurrent and consequently conservative: $q_t(x, E) = 1, x \in E$, where $\{q_t; t > 0\}$ denotes the transition function of Y . Y is m -symmetric and its Dirichlet form $(\mathcal{E}^Y, \mathcal{F}^Y)$ on $L^2(\mathbb{R}^1, m)$ is given by

$$\begin{cases} \mathcal{E}^Y(u, v) = \frac{1}{2} \int_{\mathbb{R}^1} u'(x)v'(x)m(x) dx, \\ \mathcal{F}^Y = \{u \in L^2(\mathbb{R}^1; m) : u \text{ is absolutely} \\ \text{continuous and } \mathcal{E}^Y(u, u) < \infty\} (=H^1(\mathbb{R}^1)). \end{cases}$$

For $u \in C_0^1(\mathbb{R}^1)$, $\mathcal{E}^Y(u, \frac{u}{m})$ is seen to be equal to $\frac{1}{2} \int_{\mathbb{R}^1} ((u')^2 - 2\lambda b u' u) dx$ and so

$$\mathcal{E}^Y\left(u, \frac{u}{m}\right) = \frac{1}{2} \left(\int_{\mathbb{R}^1} (u')^2 dx + \lambda \int_{\mathbb{R}^1} b' u^2 dx \right).$$

There is a finite interval $I \subset \mathbb{R}^1$ where b' is strictly negative. Choose $u_0 \in C_0^1(\mathbb{R}^1)$ not identically zero and with support being contained in I . We can then make a choice of $\lambda > 0$ such that the right-hand side of the above equation is negative for $u = u_0$.

Since q_t maps $L^2(\mathbb{R}^1; m)$ into $\mathcal{F}^Y \subset C(\mathbb{R}^1)$, $q_t(x, \cdot)$ is absolutely continuous with respect to m and hence with respect to the Lebesgue measure for each $x \in \mathbb{R}^1$. Denote by $q_t(x, y)$ its density with respect to the Lebesgue measure so that $\int_{\mathbb{R}^1} q_t(x, y) dy = 1, x \in \mathbb{R}^1$, with

$$(3.11) \quad q_t(y, x) = m(x)q_t(x, y) \frac{1}{m(y)}.$$

We know that the left-hand side of the above equation equals

$$\lim_{t \downarrow 0} \frac{1}{t} \int_{\mathbb{R}^1} (u(x) - q_t u(x)) \frac{u(x)}{m(x)} m(x) dx = \lim_{t \downarrow 0} \frac{1}{t} \int_{\mathbb{R}^1} (u(x) - q_t u(x)) u(x) dx$$

and so, for $k(x, y) = q_{t_0}(x, y)$ with a sufficiently small $t_0 > 0$,

$$\eta(u_0, u_0) = - \int_{\mathbb{R}^1} \left[\int_{\mathbb{R}^1} (u_0(y) - u_0(x)) k(x, y) dy \right] u_0(x) dx < 0.$$

Equality (3.10) follows from (3.11).

4. Associated Hunt process and martingale problem. Let (η, \mathcal{F}) be a regular lower bounded semi-Dirichlet form on $L^2(E; m)$ as is defined in Section 1. For the symmetrization $\tilde{\eta}$, $(\tilde{\eta}_{\beta_0}, \mathcal{F})$ is then a closed symmetric form on $L^2(E; m)$ but not necessarily a symmetric Dirichlet form. A symmetric Dirichlet form \mathcal{E} on $L^2(E; m)$ with domain \mathcal{F} will be called a *reference (symmetric Dirichlet) form* of η if, for each fixed $\alpha > \beta_0$,

$$(4.1) \quad c_1 \mathcal{E}_1(u, u) \leq \eta_\alpha(u, u) \leq c_2 \mathcal{E}_1(u, u), \quad u \in \mathcal{F},$$

for some positive c_1, c_2 independent of $u \in \mathcal{F}$. \mathcal{E} is then a regular Dirichlet form. In what follows, we assume that η admits a reference form \mathcal{E} . This assumption is really unnecessary (cf. [16, 19]) but convenient to simplify some arguments. The regular lower bounded semi-Dirichlet form (η, \mathcal{F}^0) constructed in Section 2 from a kernel k satisfying (2.1)–(2.4) has a reference form $(\mathcal{E}, \mathcal{F}^0)$ defined right after (1.4).

In formulating an association of a Hunt process with η , Carrillo Menendez adopted a functional capacity theorem due to Ancona [2]. More specifically, denote by \mathcal{O} the family of all open sets $A \subset E$ with $\mathcal{L}_A = \{u \in \mathcal{F} : u \geq 1 \text{ } m\text{-a.e. on } A\} \neq \emptyset$. Fix $\alpha > \beta_0$ and, for $A \in \mathcal{O}$, let e_A be the η_α -projection of 0 on \mathcal{L}_A in Stampacchia's sense [21] (cf. [17], Theorem 2.6):

$$(4.2) \quad e_A \in \mathcal{L}_A, \quad \eta_\alpha(e_A, w) \geq \eta_\alpha(e_A, e_A) \quad \text{for any } w \in \mathcal{L}_A.$$

A set $N \subset E$ is called η -polar if there exist decreasing $A_n \in \mathcal{O}$ containing N such that e_{A_n} is η_α -convergent to 0 as $n \rightarrow \infty$. A numerical function u on E is called η -quasi-continuous if there exist decreasing $A_n \in \mathcal{O}$ such that e_{A_n} is η_α -convergent to 0 as $n \rightarrow \infty$ and $u|_{E \setminus A_n}$ is continuous for each n .

The capacity Cap for the reference form \mathcal{E} is defined by

$$\text{Cap}(A) = \inf \{ \mathcal{E}_1(u, u) : u \in \mathcal{L}_A \}, \quad A \in \mathcal{O}.$$

It then follows from (4.1) that

$$(4.3) \quad c_1 \text{Cap}(A) \leq \eta_\alpha(e_A, e_A) \leq c_2 K_\alpha^2 \text{Cap}(A), \quad A \in \mathcal{O},$$

$$K_\alpha = K + \frac{\alpha}{\alpha - \beta_0},$$

because (4.2) and (B.2) imply $\eta_\alpha(e_A, e_A) \leq K_\alpha^2 \eta_\alpha(w, w), w \in \mathcal{L}_A$. Equation (4.3) means that a set N is η -polar iff it is \mathcal{E} -polar in the sense that $\text{Cap}(N) = 0$, and a function u is η -quasi-continuous iff it is \mathcal{E} -quasi-continuous

in the sense that there exist decreasing $A_n \in \mathcal{O}$ with $\text{Cap}(A_n) \downarrow 0$ as $n \rightarrow \infty$ and $u|_{E \setminus A_n}$ is continuous for each n . Every element of \mathcal{F} admits its η -quasi-continuous m -version. If $\{u_n\} \subset \mathcal{F}$ is η_α -convergent to $u \in \mathcal{F}$ and if each u_n is η -quasi-continuous, then (4.1) implies that a subsequence of $\{u_n\}$ converges η -q.e., namely, outside some η -polar set, to an η -quasi-continuous version of u . We shall occasionally drop η from the terms η -polar, η -q.e. and η -quasi-continuity for simplicity.

Recall that the L^2 -resolvent $\{G_\alpha; \alpha > \beta_0\}$ associated with η determines the resolvent $\{G_\alpha; \alpha > 0\}$ on $L^\infty(E; m)$ with $\|G_\alpha f\|_\infty \leq \frac{1}{\alpha} \|f\|_\infty$, $\alpha > 0$, $f \in L^\infty(E; m)$.

LEMMA 4.1. *Suppose $G_\beta f$ admits a quasi-continuous m -version $R_\beta f$ for a fixed $\beta > \beta_0$ and for every bounded Borel $f \in L^2(E; m)$. Then, for any α with $0 < \alpha \leq \beta_0$ and for any bounded Borel $f \in L^2(E; m)$,*

$$R_\alpha f(x) = \sum_{k=1}^{\infty} (\beta - \alpha)^{k-1} R_\beta^k f(x)$$

converges q.e. and defines a quasi-continuous m -version of $G_\alpha f$. Further the resolvent equation

$$R_\alpha f - R_\beta f + (\alpha - \beta) R_\alpha R_\beta f = 0$$

holds q.e. for any bounded Borel $f \in L^2(E; m)$.

PROOF. Choose a regular nest $\{F_\ell\}$ so that $R_\beta^k f \in C(\{F_\ell\})$ for $k \geq 1$. Define $v_n(x) = \sum_{k=1}^n (\beta - \alpha)^{k-1} R_\beta^k f(x)$. By the resolvent equation for $\{G_\alpha; \alpha > 0\}$, we have

$$G_\alpha f = v_n + (\beta - \alpha)^n G_\beta^n G_\alpha f.$$

The L^∞ -norm of the second term of the right-hand side is dominated by $\frac{1}{\alpha} (\frac{\beta - \alpha}{\beta})^n \|f\|_\infty$, which tends to 0 as $n \rightarrow \infty$. Therefore, $\{v_n\}$ is convergent uniformly on each set F_ℓ to a quasi-continuous version of $G_\alpha f$. The resolvent equation is clear. \square

THEOREM 4.1. *There exist a Borel η -polar set $N_0 \subset E$ and a Hunt process $X = (X_t, P_x)$ on $E \setminus N_0$ which is properly associated with (η, \mathcal{F}) in the sense that $R_\alpha f$ is a quasi continuous version of $G_\alpha f$ for any $\alpha > 0$ and any bounded Borel $f \in L^2(E; m)$. Here R_α is the resolvent of X and G_α is the resolvent associated with η .*

This theorem was proved in [8] first by assuming that $\beta_0 = 0$ and then reducing the situation to this case. Actually the proof can be carried out without such a reduction. Indeed, after constructing the kernel \tilde{V}_λ of [8],

Proposition II.2.1, for every rational $\lambda > \beta_0$ ([8], Proposition II.2.2) can be shown first for every rational $\lambda > \beta_0$, and then for every $0 < \lambda \leq \beta_0$ by using Lemma 4.1. The rest of the arguments in [8] then works in getting to Theorem 4.1.

Our next concern will be exceptional sets and fine continuity for the Hunt process $X = (X_t, P_x)$ appearing in Theorem 4.1. Denote by $\mathcal{B}(E)$ the family of all Borel sets of E . For $B \in \mathcal{B}(E)$, we let

$$\sigma_B = \inf\{t > 0 : X_t \in B\}, \quad \hat{\sigma}_B = \inf\{t > 0 : X_{t-} \in B\}, \quad \inf \emptyset = \infty.$$

$A \in \mathcal{B}(E)$ is called *X-invariant* if

$$P_x(\sigma_{E \setminus A} \wedge \hat{\sigma}_{E \setminus A} < \infty) = 0 \quad \forall x \in A.$$

$N \in \mathcal{B}(E)$ is called *properly exceptional* (with respect to X) if $m(N) = 0$ and $E \setminus N$ is X -invariant.

A set $N \subset E$ is called *m-polar* if there exists $N_1 \supset N, N_1 \in \mathcal{B}(E)$ such that $P_m(\sigma_{N_1} < \infty) = 0$. Any properly exceptional set is *m-polar*.

THEOREM 4.2.

- (i) For $A \in \mathcal{O}$, the function p_A^α defined by $p_A^\alpha(x) = E_x[e^{-\alpha\sigma_A}], x \in E \setminus N_0$, is a quasi-continuous version of $e_A, \alpha > \beta_0$.
- (ii) For any η -polar set B , there exists a Borel properly exceptional set N containing $N_0 \cup B$.
- (iii) If u is η -quasi-continuous, then there exists a Borel properly exceptional set $N \supset N_0$ such that, for any $x \in E \setminus N$,

$$(4.4) \quad P_x \left(\lim_{t' \downarrow t} u(X_{t'}) = u(X_t) \quad \forall t \geq 0 \quad \text{and} \quad \lim_{t' \uparrow t} u(X_{t'}) = u(X_{t-}) \quad \forall t \in (0, \zeta) \right) = 1,$$

where ζ is the lifetime of X . In particular, u is finely continuous with respect to the restricted Hunt process $X|_{E \setminus N}$.

- (iv) Any X -semi-polar set is η -polar.
- (v) A set $N \subset E$ is η -polar if and only if N is *m-polar*.

PROOF. (i) A function $u \in L^2(E; m)$ is said to be α -excessive if $u \geq 0$, $\beta G_{\alpha+\beta} u \leq u, \beta > 0$. A function $u \in \mathcal{F}$ is α -excessive iff $\eta_\alpha(u, v) \geq 0$ for all nonnegative $v \in \mathcal{F}$ (cf. [16], Theorem 2.4). In particular, e_A is α -excessive and further $v = e_A \wedge p_A^\alpha$ is an α -excessive function in \mathcal{F} (cf. [16], Theorem 2.6). Hence, $\eta_\alpha(v, e_A - v) \geq 0$. Since $v \in \mathcal{L}_A$, $\eta_\alpha(e_A, e_A - v) \leq 0$ so that $v = e_A$ and $e_A \leq p_A^\alpha$. The converse inequality can be obtained as in the proof of Theorem 6.1 below by using the optional sampling theorem for a supermartingale but with time parameter set being a finite set.

Since the quasi-continuous function $\beta R_{\alpha+\beta} p_A^\alpha$ converges to p_A^α as $\beta \rightarrow \infty$ pointwise and in η_α , we get the quasi-continuity of p_A^α .

(ii) Choose a decreasing sets $A_n \in \mathcal{O}$ with $A_n \supset B$, $\text{Cap}(A_n) \rightarrow 0$, $n \rightarrow \infty$ and put $B_1 = \bigcap_n A_n$. By (4.1) and (i), $\lim_{n \rightarrow \infty} p_{A_n}^\alpha = 0$ q.e. so that

$$P_x(\sigma_{B_1} \wedge \widehat{\sigma}_{B_1} < \infty) = 0, \quad x \in E \setminus N_1,$$

for some polar set N_1 . Choose next a decreasing sets $A'_n \in \mathcal{O}$ containing $B_1 \cup N_1 \cup N_0$ with $\text{Cap}(A'_n) \rightarrow 0$, $n \rightarrow \infty$ and put $B_2 = \bigcap_n A'_n$. Then the above identity holds for $x \in E \setminus B_2$. Moreover, the above identity holds true for B_2 in place of B_1 and for some polar set N_2 in place of N_1 . Repeating this procedure, we get an increasing sequence $\{B_k\}$ of G_δ -sets which are polar sets such that

$$P_x(\sigma_{B_k} \wedge \widehat{\sigma}_{B_k} < \infty) = 0, \quad x \in E \setminus B_{k+1}.$$

It then suffices to put $N = \bigcup_k B_k$.

(iii) Choose decreasing $A_n \in \mathcal{O}$ such that $\text{Cap}(A_n) \rightarrow 0$, $n \rightarrow \infty$, and $u|_{E \setminus A_n}$ is continuous for each n . Let N be a properly exceptional set constructed in (ii) starting with this sequence $\{A_n\}$. Then, for any $x \in E \setminus N$, $\lim_{n \rightarrow \infty} p_{A_n}^\alpha(x) = 0$ and consequently $P_x(\lim_{n \rightarrow \infty} \sigma_{A_n} = \infty) = 1$, which readily implies (4.4).

(iv) We reproduce a proof by Silverstein [20]. For $B \in \mathcal{B}(E)$, consider the entry time $\sigma_B = \inf\{t \geq 0 : X_t \in B\}$ and the function $\dot{p}_B^\alpha(x) = E_x[e^{-\alpha \sigma_B}]$, $x \in E$, $\alpha > \beta_0$. Let K be a compact thin set: K admits no regular point relative to X . It suffices to show that K is η -polar.

Choose relatively compact open sets $\{G_n\}$ such that $G_n \supset \overline{G_{n+1}}$ and $\bigcap_n G_n = K$. Due to the quasi-left continuity of X , $p_{G_n}^\alpha(x) = \dot{p}_{G_n}^\alpha(x)$ then decreases to $\dot{p}_K^\alpha(x)$ as $n \rightarrow \infty$ for each $x \in E$. By (i) and (4.1) and (4.2), the sequence $\{\dot{p}_{G_n}^\alpha\}$ is \mathcal{E}_1 -bounded so that the Cesàro mean sequence f_n of its suitable subsequence is \mathcal{E}_1 -convergent. Since f_n are quasi-continuous and converges to \dot{p}_K^α pointwise as $n \rightarrow \infty$, we conclude that \dot{p}_K^α is a quasi-continuous element of \mathcal{F} . On the other hand, the quasi-continuous function $\beta R_{\alpha+\beta} \dot{p}_K^\alpha$ converges to p_K^α as $\beta \rightarrow \infty$ pointwise and in η_α so that p_K^α is also a quasi-continuous version of \dot{p}_K^α . Therefore, $p_K^\alpha = \dot{p}_K^\alpha$ q.e. and in particular K is η -polar.

(v) “only if” part follows from (ii). To show “if” part, assume that K is a compact m -polar set. Then $p_K^\alpha = 0$ m -a.e. Choose for K relatively compact open sets $\{G_n\}$ as in the proof of (iv) so that the Cesàro mean f_ℓ of a certain subsequence $\{p_{G_{n_\ell}}^\alpha\}$ is \mathcal{E}_1 -convergent to p_K^α as $\ell \rightarrow \infty$ which is now a zero element of \mathcal{F}^0 . Since $f_\ell \geq 1$ m -a.e. on G_{n_ℓ} , we have $\text{Cap}(K) \leq \text{Cap}(G_{n_\ell}) \leq \mathcal{E}_1(f_\ell, f_\ell)$ and we get $\text{Cap}(K) = 0$ by letting $\ell \rightarrow \infty$. For any Borel m -polar set N , we have $\text{Cap}(N) = \sup\{\text{Cap}(K) : K \subset N, K \text{ is compact}\} = 0$. \square

Clearly, the restriction of X outside its properly exceptional set is again a Hunt process properly associated with η .

Our final task in this section is to relate the Hunt process of Theorem 4.1 to a martingale problem.

We consider the case where η admits the expression

$$(4.5) \quad \eta(f, g) = -(\mathcal{L}f, g), \quad f \in \mathcal{D}(\mathcal{L}), g \in \mathcal{F},$$

for a operator \mathcal{L} with domain $\mathcal{D}(\mathcal{L})$ satisfying the following:

(L.1) $\mathcal{D}(\mathcal{L})$ is a linear subspace of $\mathcal{F} \cap C_0(E)$,

(L.2) \mathcal{L} is a linear operator sending $\mathcal{D}(\mathcal{L})$ into $L^2(E; m) \cap C_b(E)$,

(L.3) there exists a countable subfamily \mathcal{D}_0 of $\mathcal{D}(\mathcal{L})$ such that each $f \in \mathcal{D}(\mathcal{L})$ admits $f_n \in \mathcal{D}_0$ such that $f_n, \mathcal{L}f_n$ are uniformly bounded and converge pointwise to $f, \mathcal{L}f$, respectively, as $n \rightarrow \infty$.

We also consider an additional condition that

(L.4) there exists $f_n \in \mathcal{D}(\mathcal{L})$ such that $f_n, \mathcal{L}f_n$ are uniformly bounded and converge to 1, 0, respectively, as $n \rightarrow \infty$.

THEOREM 4.3. *Assume that η admits the expression (4.5) with \mathcal{L} satisfying conditions (L.1), (L.2), (L.3).*

(i) *There exists then a Borel properly exceptional set N containing N_0 such that, for every $f \in \mathcal{D}(\mathcal{L})$,*

$$(4.6) \quad M_t^{[f]} = f(X_t) - f(X_0) - \int_0^t (\mathcal{L}f)(X_s) ds, \quad t \geq 0,$$

is a P_x -martingale for each $x \in E \setminus N$.

(ii) *If the additional condition (L.4) is satisfied, then the Hunt process $X|_{E \setminus N}$ is conservative.*

PROOF. (i) Take $f \in \mathcal{D}(\mathcal{L})$ and $g \in L^2(E; m)$. By (4.5) and (3.2), we have, for $\alpha > \beta_0$,

$$\begin{aligned} (G_\alpha \mathcal{L}f, g) &= (\mathcal{L}f, \widehat{G}_\alpha g) = -\eta(f, \widehat{G}_\alpha g) \\ &= -\eta_\alpha(f, \widehat{G}_\alpha g) + \alpha(f, \widehat{G}_\alpha g) \\ &= -(f, g) + \alpha(G_\alpha f, g). \end{aligned}$$

Thus, $(G_\alpha \mathcal{L}f, g) = (\alpha G_\alpha f - f, g)$ holds for any $g \in \mathcal{F}$ and

$$\frac{1}{\alpha} G_\alpha (\mathcal{L}f)(x) = G_\alpha f(x) - \frac{f(x)}{\alpha}, \quad m\text{-a.e.}$$

We denote by $\{P_t; t \geq 0\}$ and $\{R_\alpha; \alpha > 0\}$ the transition function and the resolvent of X , respectively:

$$P_t h(x) = \mathbb{E}_x[h(X_t)], \quad R_\alpha h(x) = \int_0^\infty e^{-\alpha t} P_t h(x) dt.$$

Since X is properly associated with η by Theorem 4.1, we get

$$\frac{1}{\alpha} R_\alpha (\mathcal{L}f)(x) = R_\alpha f(x) - \frac{f(x)}{\alpha}, \quad \text{q.e.}$$

Hence, by virtue of Theorem 4.2(ii), there exists a Borel properly exceptional set N such that

$$\int_0^\infty e^{-\alpha t} \left(\int_0^t P_s(\mathcal{L}f)(x) ds \right) dt = \int_0^\infty e^{-\alpha t} (P_t f(x) - f(x)) dt, \quad x \in E \setminus N,$$

holds for any $\alpha \in \mathbb{Q}_+$ with $\alpha > \beta_0$ and for any $f \in \mathcal{D}_0$.

Since $P_t h(x)$ is a right continuous in $t \geq 0$ for any $h \in C_b(E)$, we get

$$(4.7) \quad P_t f(x) - f(x) = \int_0^t P_s(\mathcal{L}f)(x) ds, \quad t \geq 0, x \in E \setminus N,$$

holding for any $f \in \mathcal{D}_0$. By virtue of condition (L.3), we conclude that the equation (4.7) holds true for any $f \in \mathcal{D}(\mathcal{L})$. Equation (4.7) implies that, for any $f \in \mathcal{D}(\mathcal{L})$, the functional $M_t^{[f]}, t \geq 0$, defined by (4.6) is a mean zero, square integrable additive functional of the Hunt process $X|_{E \setminus N}$ so that it is a P_x -martingale for each $x \in E \setminus N$.

(ii) Under the additional condition (L.4), we let $n \rightarrow \infty$ in equation (4.7) with f_n in place of f arriving at $P_t 1 = 1, t \geq 0$. \square

Theorem 4.3 will enable us in the next section to relate our Hunt process to the solution of a martingale problem in a specific case.

5. Stable-like process. In this section, we consider the case that $E = \mathbb{R}^d$ and $m(dx) = dx$ is the Lebesgue measure on \mathbb{R}^d . For a positive measurable function $\alpha(x)$ defined on \mathbb{R}^d , Bass introduced the following integro-differential operator in [5] (see also [4, 6]): for $u \in C_b^2(\mathbb{R}^d)$,

$$\mathcal{L}u(x) = w(x) \int_{h \neq 0} (u(x+h) - u(x) - \nabla u(x) \cdot h \mathbf{1}_{B(1)}(h)) |h|^{-d-\alpha(x)} dh, \quad x \in \mathbb{R}^d,$$

where $w(x)$ is a function chosen so that $\mathcal{L}e^{iux} = -|u|^{\alpha(x)} e^{iux}$ and $C_b^2(\mathbb{R}^d)$ denotes the set of twice differentiable bounded functions. If α is Lipschitz continuous, bounded below by a constant which is greater than 0, and bounded above by a constant which is less than 2, then he constructed a unique strong Markov process associated with \mathcal{L} by solving the \mathcal{L} -martingale problem for every starting point $x \in \mathbb{R}^d$. Using the theory of stochastic differential equation with jumps, Tsuchiya [22] also succeeded in constructing the Markov process associated with \mathcal{L} (see also [18]). Note that the weight function $w(x)$ is given by

$$(5.1) \quad w(x) = \frac{\Gamma((1+\alpha(x))/2) \Gamma((\alpha(x)+d)/2) \sin(\pi\alpha(x)/2)}{2^{1-\alpha(x)} \pi^{d/2+1}}, \quad x \in \mathbb{R}^d$$

(see, e.g., [3]).

Put $k(x, y) = w(x)|x - y|^{-d-\alpha(x)}$, $x, y \in \mathbb{R}^d$ with $x \neq y$. Then this falls into our case when we consider the following conditions: there exist positive constants $\underline{\alpha}, \bar{\alpha}, M$ and δ so that for $x, y \in \mathbb{R}^d$,

$$(5.2) \quad \begin{aligned} &0 < \underline{\alpha} \leq \alpha(x) \leq \bar{\alpha} < 2, \bar{\alpha} < 1 + \frac{\alpha}{2} \quad \text{and} \\ &|\alpha(x) - \alpha(y)| \leq M|x - y|^\delta \quad \text{for } \delta \text{ with } 0 < \frac{1}{2}(2\bar{\alpha} - \underline{\alpha}) < \delta \leq 1. \end{aligned}$$

PROPOSITION 5.1. *Assume (5.2) holds. Then conditions (2.1)–(2.4) are satisfied by the function*

$$(5.3) \quad k(x, y) = w(x)|x - y|^{-d-\alpha(x)}, \quad x, y \in \mathbb{R}^d, x \neq y.$$

PROOF. Note first that, from equation (5.1) defining the weight $w(x)$, we easily see that there exist constants c_i ($i = 1, 2, 3$) so that for $x, y \in \mathbb{R}^d$,

$$c_1 \leq w(x) \leq c_2, \quad |w(x) - w(y)| \leq c_3|\alpha(x) - \alpha(y)|.$$

Then

$$\begin{aligned} k_s(x, y) &= \frac{1}{2}(w(x)|x - y|^{-d-\alpha(x)} + w(y)|x - y|^{-d-\alpha(y)}) \\ &\leq \begin{cases} M|x - y|^{-d-\bar{\alpha}}, & |x - y| \leq 1, \\ M|x - y|^{-d-\underline{\alpha}}, & |x - y| > 1. \end{cases} \end{aligned}$$

This and the condition $0 < \underline{\alpha} \leq \bar{\alpha} < 2$ imply that condition (2.1) is fulfilled because the function M_s in it is bounded. Condition (2.2) is also valid as $|k_a(x, y)| \leq k_s(x, y)$.

On the other hand, since

$$\begin{aligned} k_a(x, y) &= w(x)|x - y|^{-d-\alpha(x)} - w(y)|x - y|^{-d-\alpha(y)} \\ &= (w(x) - w(y))|x - y|^{-d-\alpha(x)} \\ &\quad + w(y)|x - y|^{-d}(|x - y|^{-\alpha(x)} - |x - y|^{-\alpha(y)}) \end{aligned}$$

and

$$|x - y|^{-\alpha(x)} - |x - y|^{-\alpha(y)} = \int_{\alpha(y)}^{\alpha(x)} |x - y|^{-u} \frac{1}{\ln|x - y|^{-1}} du,$$

we see that for $|x - y| < 1$,

$$\begin{aligned} |k_a(x, y)| &\leq |w(x) - w(y)| \cdot |x - y|^{-d-\alpha(x)} \\ &\quad + w(y)|x - y|^{-d}|\alpha(x) - \alpha(y)| \cdot |x - y|^{-(\alpha(x) \vee \alpha(y))} \frac{1}{\ln|x - y|^{-1}} \end{aligned}$$

$$\begin{aligned} &\leq M \left(|x-y|^{-d-\bar{\alpha}+\delta} + |x-y|^{-d-\bar{\alpha}+\delta} \frac{1}{\ln|x-y|^{-1}} \right) \\ &\leq M' |x-y|^{-d-\bar{\alpha}+\delta} \frac{1}{\ln|x-y|^{-1}}. \end{aligned}$$

So if γ satisfies

$$\gamma(d + \bar{\alpha} - \delta) - (d - 1) < 1,$$

then condition (2.3) holds. As for condition (2.4), note that

$$k_s(x, y) \geq M' |x-y|^{-d-\alpha}, \quad |x-y| < 1.$$

So, (2.4) is valid when

$$(d + \bar{\alpha} - \delta)(2 - \gamma) < d + \underline{\alpha}.$$

Therefore, conditions (2.3) and (2.4) hold provided that γ satisfies

$$\frac{d + 2\bar{\alpha} - 2\delta - \underline{\alpha}}{d + \bar{\alpha} - \delta} < \gamma < \frac{d}{d + \bar{\alpha} - \delta}. \quad \square$$

Let (η, \mathcal{F}^0) be the regular lower bounded semi-Dirichlet form on $L^2(\mathbb{R}^d)$ associated with the kernel (5.3) satisfying (5.2) according to Theorem 2.1. Let $X = (X_t, P_x)$ be the Hunt process on \mathbb{R}^d properly associated with (η, \mathcal{F}) by Theorem 4.1.

Define a linear operator \mathcal{L} by

$$(5.4) \quad \begin{cases} \mathcal{D}(\mathcal{L}) = C_0^2(\mathbb{R}^d), \\ \mathcal{L}u(x) = \int_{h \neq 0} (u(x+h) - u(x) - \nabla u(x) \cdot h \mathbf{1}_{B_1(0)}(h)) \frac{w(x) dh}{|h|^{d+\alpha(x)}}, \\ x \in \mathbb{R}^d. \end{cases}$$

$C_0^2(\mathbb{R}^d)$ is a linear subspace of $\mathcal{F}^0 \cap C_0(\mathbb{R}^d)$ and, by condition (5.2), we can see that \mathcal{L} maps $C_0^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^d) \cap C_b(\mathbb{R}^d)$. As any continuously differentiable function and its derivatives can be simultaneously approximated by polynomials and their derivatives uniformly on each rectangles (cf. [9], Chapter II), conditions (L.1), (L.2), (L.3) in the preceding section on \mathcal{L} are fulfilled. We can easily verify that the present \mathcal{L} satisfies condition (L.4) as well.

Since the vector valued function $hw(x)\mathbf{1}_{B_1(0)}(h)|h|^{-d-\alpha(x)}$ is odd with respect to the variable h for each $x \in \mathbb{R}^d$, we get for $u \in C_0^2(\mathbb{R}^d)$,

$$\begin{aligned} \eta^n(u, v) &= - \int \int_{|x-y| > 1/n} (u(y) - u(x))v(x) \frac{w(x)}{|x-y|^{d+\alpha(x)}} dx dy \\ &= - \int \int_{|h| > 1/n} (u(x+h) - u(x))v(x) \frac{w(x)}{|h|^{d+\alpha(x)}} dx dh \end{aligned}$$

$$\begin{aligned}
&= - \int \int_{|h| > 1/n} (u(x+h) - u(x) - \nabla u(x) \cdot h \mathbf{1}_{B_1(0)}(h)) v(x) \\
&\quad \times \frac{w(x)}{|h|^{d+\alpha(x)}} dx dh.
\end{aligned}$$

By letting $n \rightarrow \infty$, we have

$$\eta(u, v) = -(\mathcal{L}u, v),$$

that is, η is related to \mathcal{L} by (4.5).

By virtue of Theorem 4.3, there exists a Borel properly exceptional set $N \subset \mathbb{R}^d$ so that $X|_{\mathbb{R}^d \setminus N}$ is conservative and, for each $x \in \mathbb{R}^d \setminus N$,

$$M_t^{[f]} = f(X_t) - f(X_0) - \int_0^t (\mathcal{L}f)(X_s) ds, \quad t \geq 0,$$

is a martingale under P_x for every $f \in C_0^2(\mathbb{R}^d)$. Approximating $f \in C_b^2(\mathbb{R}^d)$ by a uniformly bounded sequence $\{f_n\} \subset C_0^2(\mathbb{R}^d)$ such that $\{\mathcal{L}f_n\}$ is uniformly bounded and convergent to $\mathcal{L}f$, we see that (4.6) remains valid for $f \in C_b^2(\mathbb{R}^d)$ and $M_t^{[f]}$ is still a martingale under \mathbb{P}_x for $x \in \mathbb{R}^d \setminus N$. For each $x \in \mathbb{R}^d \setminus N$, the measure \mathbb{P}_x is thus a solution to the martingale problem for the operator \mathcal{L} of (5.4) starting at x so that \mathbb{P}_x coincides with the law constructed by Bass [5] because of the uniqueness also due to [5].

REMARK 5.1. Let

$$(5.5) \quad k^*(x, y) = \frac{w(y)}{|x - y|^{d+\alpha(y)}}, \quad x, y \in \mathbb{R}^d, x \neq y.$$

Under condition (5.2), the form η^* corresponding to the kernel k^* is a regular lower bounded semi-Dirichlet form on $L^2(\mathbb{R}^d)$ by virtue of Proposition 5.1 and Corollary 2.1. By Theorem 4.1, η^* admits a properly associated Hunt process X^* on \mathbb{R}^d . Furthermore, we can have an explicit expression $\eta^*(u, v) = -(\mathcal{L}^*u, v)$ for $u \in C_0^2(\mathbb{R}^d)$ and $v \in \mathcal{F}^0$ with

$$\begin{aligned}
\mathcal{L}^*u(x) &= \int_{h \neq 0} (u(x+h) - u(x) - \nabla u(x) \cdot h \mathbf{1}_{B_1(0)}(h)) \frac{w(x+h) dh}{|h|^{d+\alpha(x+h)}} \\
&\quad + \frac{1}{2} \int_{0 < |h| < 1} \nabla u(x) \cdot h \left(\frac{w(x+h)}{|h|^{d+\alpha(x+h)}} - \frac{w(x-h)}{|h|^{d+\alpha(x-h)}} \right) dh, \quad x \in \mathbb{R}^d.
\end{aligned}$$

In a lower order case as is considered in Section 3, both \mathcal{L} and \mathcal{L}^* admit simpler expressions (3.7) and $\mathcal{L}^* - K$ is a formal adjoint of \mathcal{L} for a function K defined by (3.9).

6. Associated Hunt processes on open subsets and on their closures. We make the same assumptions on E, m, k as in Section 2. Let D be an arbitrary open subset of E and \overline{D} be the closure of D , m_D is defined to be $m_D(B) = m(B \cap D)$, $B \in \mathcal{B}(E)$ and $(u, v)_D$ denotes the inner product of $L^2(D, m_D)$

($=L^2(\overline{D}, m_D)$). Consider the related function spaces $C_0^{\text{lip}}(\overline{D})$ and $C_0^{\text{lip}}(D)$ introduced in Section 1. Define

$$(6.1) \quad \begin{cases} \mathcal{E}_D(u, v) := \int \int_{D \times D \setminus \text{diag}} (u(y) - u(x))(v(y) - v(x)) \\ \quad \times k_s(x, y) m_D(dx) m_D(dy), \\ \mathcal{F}_D^r = \{u \in L^2(D; m_D) : u \text{ is Borel measurable and } \mathcal{E}_D(u, u) < \infty\}, \end{cases}$$

and let $\mathcal{F}_{\overline{D}}$ and \mathcal{F}_D^0 be the $\mathcal{E}_{D,1}$ -closures of $C_0^{\text{lip}}(\overline{D})$ and $C_0^{\text{lip}}(D)$ in \mathcal{F}_D^r , respectively. $(\mathcal{E}_D, \mathcal{F}_{\overline{D}})$ [resp., $(\mathcal{E}_D^0, \mathcal{F}_D^0)$] is a regular symmetric Dirichlet form on $L^2(\overline{D}; m_D)$ [resp., $L^2(D; m_D)$] where \mathcal{E}_D^0 denotes the restriction of \mathcal{E}_D to $\mathcal{F}_D^0 \times \mathcal{F}_D^0$. Furthermore, in view of [13], Theorem 4.4.3, we have the identity

$$(6.2) \quad \mathcal{F}_D^0 = \{u \in \mathcal{F}_{\overline{D}} : \tilde{u} = 0, \mathcal{E}_D\text{-q.e. on } \partial D\},$$

where \tilde{u} denotes an \mathcal{E}_D -quasi continuous version of $u \in \mathcal{F}_{\overline{D}}$. We keep in mind that a subset of D is polar for $(\mathcal{E}_D, \mathcal{F}_D^0)$ iff so it is for $(\mathcal{E}_D, \mathcal{F}_{\overline{D}})$, and the restriction to D of a quasi continuous function with respect to the latter is quasi-continuous with respect to the former.

Now define for $u \in C_0^{\text{lip}}(\overline{D})$ and $n \in \mathbb{N}$

$$(6.3) \quad \mathcal{L}_D^n u(x) := \int_{\{y \in D : d(x, y) > 1/n\}} (u(y) - u(x)) k(x, y) m_D(dy), \quad x \in D.$$

Then, just as in Proposition 2.1 and Theorem 2.1 of Section 2, we conclude that the finite limit

$$(6.4) \quad \eta_D(u, v) = - \lim_{n \rightarrow \infty} \int_D \mathcal{L}_D^n u(x) v(x) m_D(dx) \quad \text{for } u, v \in C_0^{\text{lip}}(\overline{D})$$

exists, η_D extends to $\mathcal{F}_{\overline{D}} \times \mathcal{F}_{\overline{D}}$ and $(\eta_D, \mathcal{F}_{\overline{D}})$ becomes a regular lower bounded semi-Dirichlet form on $L^2(\overline{D}; m_D)$ possessing $(\mathcal{E}_D, \mathcal{F}_{\overline{D}})$ as its reference symmetric Dirichlet form. In parallel with $(\eta_D, \mathcal{F}_{\overline{D}})$, the space $(\eta_D^0, \mathcal{F}_D^0)$ becomes a regular lower bounded semi-Dirichlet form on $L^2(D; m_D)$ possessing $(\mathcal{E}_D^0, \mathcal{F}_D^0)$ as its reference symmetric Dirichlet form. Here η_D^0 is the restriction of η_D to $\mathcal{F}_D^0 \times \mathcal{F}_D^0$.

Let $X^{\overline{D}} = (X_t, P_x)$ be a Hunt process on \overline{D} properly associated with the form $(\eta_D, \mathcal{F}_{\overline{D}})$ on $L^2(\overline{D}; m_D)$. Denote by $X^{D,0} = (X_t^{D,0}, P_x)$ the part process of $X^{\overline{D}}$ on D , namely, $X_t^{D,0}$ is obtained from X_t by killing upon hitting the boundary ∂D :

$$X_t^{D,0} = X_t, \quad t < \sigma_{\partial D}; \quad X_t^{D,0} = \Delta, \quad t \geq \sigma_{\partial D},$$

$X^{D,0}$ is a Hunt process with state space D .

THEOREM 6.1. *The part process $X^{D,0}$ of $X^{\overline{D}}$ on D is properly associated with the regular lower bounded semi-Dirichlet form $(\eta_D^0, \mathcal{F}_D^0)$ on $L^2(D; m_D)$.*

PROOF. Let $\{R_\alpha; \alpha > 0\}$ be the resolvent of $X^{\bar{D}}$. σ will denote the hitting time of ∂D by $X^{\bar{D}}: \sigma = \sigma_{\partial D}$. Put, for $\alpha > 0$ and $x \in \bar{D}$,

$$R_\alpha^{D,0} f(x) = E_x \left[\int_0^\sigma e^{-\alpha t} f(X_t) dt \right],$$

$$H_\alpha^{\partial D} u(x) = E_x [e^{-\alpha \sigma} u(X_\sigma)], \quad x \in \bar{D}.$$

$\{R_\alpha^{D,0}|_D; \alpha > 0\}$ is the resolvent of the part process $X^{D,0}$ of $X^{\bar{D}}$ on D .

We need to prove that, for any $\alpha > \beta_0$ and any $f \in \mathcal{B}(\bar{D}) \cap L^2(\bar{D}, m_D)$,

$$(6.5) \quad \begin{aligned} & R_\alpha^{D,0} f \text{ is } \eta_D^0\text{-quasi-continuous,} \\ & R_\alpha^{D,0} f \in \mathcal{F}_D^0, \quad \eta_{D,\alpha}^0(R_\alpha^{D,0} f, v) = (f, v)_D \quad \text{for any } v \in \mathcal{F}_D^0. \end{aligned}$$

We denote by \mathcal{G} the space appearing in the right-hand side of (6.2). Notice that \mathcal{E}_D -q.e. (resp., \mathcal{E}_D -quasi-continuity) is now a synonym of η_D -q.e. (resp., η_D -quasi-continuity). As the set of points of ∂D that are irregular for ∂D is known to be semi-polar, we have $P_x(\sigma = 0) = 1$ and so $R_\alpha^{D,0} f(x) = 0$ for η_D -q.e. $x \in \partial D$ owing to Theorem 4.2(iv). Since

$$\begin{aligned} & R_\alpha f \text{ is } \eta_D\text{-quasi-continuous,} \\ & R_\alpha f \in \mathcal{F}_{\bar{D}}, \quad \eta_{D,\alpha}(R_\alpha f, v) = (f, v)_D \quad \text{for any } v \in \mathcal{F}_{\bar{D}} \end{aligned}$$

and

$$(6.6) \quad R_\alpha f(x) = R_\alpha^{D,0} f(x) + H_\alpha^{\partial D} R_\alpha f(x), \quad x \in \bar{D},$$

we see that, for the proof of (6.5), it is enough to show that

$$(6.7) \quad \begin{aligned} & H_\alpha^{\partial D} R_\alpha f \text{ is } \eta_D\text{-quasi-continuous,} \\ & H_\alpha^{\partial D} R_\alpha f \in \mathcal{F}_{\bar{D}}, \quad \eta_{D,\alpha}(H_\alpha^{\partial D} R_\alpha f, v) = 0 \quad \text{for any } v \in \mathcal{G}. \end{aligned}$$

To this end, we fix $\alpha > \beta_0$, $f \in \mathcal{B}_+(\bar{D}) \cap L^2(\bar{D}; m_D)$ and put $u = R_\alpha f$. Consider a closed convex subset of $\mathcal{F}_{\bar{D}}$ defined by

$$\mathcal{L}_{u,\partial D} = \{v \in \mathcal{F}_{\bar{D}}, \tilde{v} \geq \tilde{u} \text{ q.e. on } \partial D\}.$$

Let u_α be the $\eta_{D,\alpha}$ -projection of 0 on $\mathcal{L}_{u,\partial D}$:

$$u_\alpha \in \mathcal{L}_{u,\partial D}, \quad \eta_{D,\alpha}(u_\alpha, v - u_\alpha) \geq 0, \quad \text{for any } v \in \mathcal{L}_{u,\partial D}.$$

Both u and u_α are α -excessive elements of $\mathcal{F}_{\bar{D}}$. By making use of the function $v = u_\alpha \wedge u$ as in the proof of Proposition 3.1(i), we readily get

$$(6.8) \quad \tilde{u}_\alpha = u \text{ q.e. on } \partial D, \quad \eta_{D,\alpha}(u_\alpha, v) = 0 \quad \text{for any } v \in \mathcal{G}.$$

Finally, we prove that

$$(6.9) \quad H_\alpha^{\partial D} u \text{ is } \eta_D\text{-quasi continuous,} \quad H_\alpha^{\partial D} u = u_\alpha,$$

which leads us to the desired property (6.7). By (6.6), $H_\alpha^{\partial D}u$ is an α -excessive function dominated by $u \in \mathcal{F}_{\bar{D}}$ so that $H_\alpha^{\partial D}u$ is a quasi-continuous element of $\mathcal{F}_{\bar{D}}$. Further $H_\alpha^{\partial D}u = u$ q.e. on ∂D by (6.6) and an observation made preceding it. Let $v = H_\alpha^{\partial D}u \wedge u_\alpha$. Then $\tilde{v} = H_\alpha^{\partial D}u \wedge \tilde{u}_\alpha = u$ q.e. on ∂D so that $\eta_{D,\alpha}(u_\alpha, u_\alpha - v) = 0$ by (6.8). On the other hand, v is α -excessive and so $\eta_{D,\alpha}(v, u_\alpha - v) \geq 0$. Consequently, $\eta_\alpha(u_\alpha - v, u_\alpha - v) \leq 0$ and we get the inequality $u_\alpha \leq H_\alpha^{\partial D}u$.

To get the converse inequality, consider a bounded nonnegative Borel function h on D with $\int_D h dm = 1$. Denote by $\{p_t; t \geq 0\}$ the transition function of $X^{\bar{D}}$. We choose a Borel measurable quasi-continuous version \tilde{u}_α of $u_\alpha \in \mathcal{F}_{\bar{D}}$. We set $\tilde{u}_\alpha(\Delta) = 0$ for the cemetery Δ of $X^{\bar{D}}$. Since u_α is α -excessive, $e^{-\alpha t} p_t \tilde{u}_\alpha \leq \tilde{u}_\alpha$ m -a.e., and we can see that the process $\{Y_t = e^{-\alpha t} \tilde{u}_\alpha(X_t); t \geq 0\}$ is a right continuous positive supermartingale under $P_{h \cdot m}$ in view of Theorem 4.2(iii). For any compact set $K \subset \partial D$, we get from the optional sampling theorem and (6.8),

$$\begin{aligned} E_{h \cdot m}[Y_{\sigma_K}] &= E_{h \cdot m}[e^{-\alpha \sigma_K} \tilde{u}_\alpha(X_{\sigma_K})] \\ &= E_{h \cdot m}[e^{-\alpha \sigma_K} u(X_{\sigma_K})] \leq E_{h \cdot m}[Y_0] \\ &= (h, u_\alpha)_D. \end{aligned}$$

By choosing K such that $\sigma_K \downarrow \sigma$ $P_{h \cdot m}$ -a.e., we obtain $(h, H_\alpha^{\partial D}u)_D \leq (h, u_\alpha)_D$ and $H_\alpha^{\partial D}u \leq u_\alpha$. \square

As a preparation for the next lemma, we take any open set $G \subset D$ and denote by m_G the restriction of m to G . Let \mathcal{F}_G^0 be the $\mathcal{E}_{D,1}$ -closure of $C_0^{\text{lip}}(G)$ in \mathcal{F}_D^r and η_G^0 be the restriction of η_D to $\mathcal{F}_G^0 \times \mathcal{F}_G^0$. Then, just as above,

$$\mathcal{F}_G^0 = \{u \in \mathcal{F}_{\bar{D}} : \tilde{u} = 0 \text{ } \mathcal{E}_D \text{ q.e. on } \bar{D} \setminus G\}$$

and $(\eta_G^0, \mathcal{F}_G^0)$ becomes a regular lower bounded semi-Dirichlet form on $L^2(G; m_G)$ with which the part process $X^{G,0}$ of $X^{\bar{D}}$ on G is properly associated. The resolvent of $X^{G,0}$ will be denoted by $R_\alpha^{G,0}$.

Define

$$H_\alpha^{\bar{D} \setminus G} u(x) = E_x[e^{-\alpha \sigma_{\bar{D} \setminus G}} u(X_{\sigma_{\bar{D} \setminus G}})], \quad x \in \bar{D}.$$

As (6.7), we have, for $u = R_\alpha f$, $f \in \mathcal{B}(\bar{D}) \cap L^2(\bar{D}; m_D)$, $\alpha > \beta_0$,

$$\begin{aligned} (6.10) \quad & H_\alpha^{\bar{D} \setminus G} u \text{ is } \eta_D\text{-quasi-continuous,} \\ & H_\alpha^{\bar{D} \setminus G} u \in \mathcal{F}_{\bar{D}}, \quad \eta_{D,\alpha}(H_\alpha^{\bar{D} \setminus G} u, v) = 0 \quad \text{for any } v \in \mathcal{F}_G^0, \end{aligned}$$

and the bound $\eta_{D,\alpha}(H_\alpha^{\bar{D} \setminus G} u, H_\alpha^{\bar{D} \setminus G} u) \leq \eta_{D,\alpha}(u, u)$. We can easily see that (6.10) holds true for any $u \in \mathcal{F}^{\bar{D}} \cap C_0(\bar{D})$ where $C_0(\bar{D})$ denotes the

restrictions to \overline{D} of functions in $C_0(E)$. In fact, by the resolvent equation, (6.10) is true for $R_\beta u$, $\beta > \beta_0$, in place of u . Since $\{\beta_n R_{\beta_n} u\}$ converges to u pointwise as well as in $\eta_{D,\alpha}$ -metric as $\beta_n \rightarrow \infty$, so does the sequence $\{\beta_n H_\alpha^{\overline{D}\setminus G} R_{\beta_n} u\}$, arriving at the validity of (6.10) for such u .

LEMMA 6.1. *Let G be a relatively compact open set with $\overline{G} \subset D$. Then for any $v \in \mathcal{F}^{\overline{D}} \cap C_0(\overline{D})$ with $\text{supp}[v] \subset \overline{D} \setminus \overline{G}$, it follows for $\alpha > \beta_0$ that*

$$(6.11) \quad E_x[e^{-\alpha\tau_G} v(X_{\tau_G})] = R_\alpha^{G,0} g_v(x) \quad \text{for } q.e. \ x \in G,$$

where $\tau_G = \sigma_{\overline{D}\setminus G} \wedge \zeta$ is the first leaving time from G and g_v is a function given by

$$(6.12) \quad g_v(x) = 1_G(x) \int_{\overline{D}\setminus\overline{G}} k(x,y)v(y)m_D(dy), \quad x \in \overline{D}.$$

PROOF. Take any $u \in \mathcal{F}^{\overline{D}} \cap C_0(\overline{D})$ such that $\text{supp}[u] \subset G$. From (6.3) and (6.4), we then have

$$(6.13) \quad \eta_D(u, v) = - \int_{G \times (\overline{D}\setminus\overline{G})} u(y)v(x)k(x,y)m_D(dx)m_D(dy).$$

We can now proceed as in [13], page 163. The function g_v defined by (6.12) belongs to $L^2(G; m_G)$ on account of condition (2.1) on the kernel k . Therefore, we obtain from (6.13)

$$\begin{aligned} \eta_{G,\alpha}^0(R_\alpha^{G,0} g_v, u) &= \int_G g_v(x)u(x)m_G(dx) \\ &= \int_{G \times (\overline{D}\setminus\overline{G})} u(x)v(y)k(x,y)m_D(dx)m_D(dy) \\ &= -\eta_D(v, u) = -\eta_{D,\alpha}(v, u) \\ &= -\eta_{G,\alpha}^0(v - H_\alpha^{\overline{D}\setminus G} v, u), \quad \alpha > \beta_0, \end{aligned}$$

the last identity being a consequence of (6.10). Since $\mathcal{F}^{\overline{D}} \cap C_0(G)$ is $\eta_{G,\alpha}^0$ -dense in \mathcal{F}_G^0 , we get

$$H_\alpha^{\overline{D}\setminus G} v(x) = H_\alpha^{\overline{D}\setminus G} v(x) - v(x) = R_\alpha^{G,0} g_v(x) \quad \text{for } m_G\text{-a.e. on } G.$$

We then obtain (6.11) because $H_\alpha^{\overline{D}\setminus G} v$ and $R_\alpha^{G,0} g_v$ are η_G^0 -quasi-continuous by (6.10). \square

THEOREM 6.2.

(i) $X^{\overline{D}} = (X_t, P_x)$ admits no jump from D to ∂D :

$$(6.14) \quad P_x(X_{t-} \in D, X_t \in \partial D \text{ for some } t > 0) = 0 \quad \text{for } q.e. \ x \in D.$$

(ii) If D is relatively compact, then $X^{\bar{D}}$ is conservative: denoting by ζ the lifetime of $X^{\bar{D}}$,

$$(6.15) \quad P_x(\zeta = \infty) = 1 \quad \text{for q.e. } x \in \bar{D}.$$

(iii) If D is relatively compact, then $X^{D,0} = (X_t^{D,0}, P_x)$ admits no killing inside D : denoting by ζ^0 the lifetime of $X^{D,0}$,

$$(6.16) \quad P_x(X_{\zeta^0-}^{D,0} \in D, \zeta^0 < \infty) = 0 \quad \text{for q.e. } x \in D.$$

PROOF. (i) For any open set G as Lemma 6.1 and any compact subset F of ∂D , we can find a uniformly bounded sequence $\{v_n\} \subset \mathcal{F}^{\bar{D}} \cap C_0(\bar{D})$ with support being contained in a common compact subset of $\bar{D} \setminus \bar{G}$ and $\lim_{n \rightarrow \infty} v_n = 1_F$. Then $g_{v_n}(x)$ are uniformly bounded and converge to $g_{1_F}(x) = 0$ as $n \rightarrow \infty$. Therefore, by letting $n \rightarrow \infty$ in (6.11) with v_n in place of v , we get $P_x(X_{\tau_G} \in F) = 0$ for q.e. $x \in G$. Since G and F are arbitrary with the stated properties, we have (6.14).

(ii) When D is relatively compact, $1 \in C_0^{\text{lip}}(\bar{D})$ so that we see from (6.3) and (6.4) that $1 \in \mathcal{F}^{\bar{D}}$ and $\eta_D(1, v) = 0$ for any $v \in \mathcal{F}^{\bar{D}}$. We have therefore, for any $\alpha > \beta_0$ and $f \in L^2(\bar{D}, m_D)$,

$$0 = \eta_D(1, \hat{G}_\alpha f) = (1, f)_D - \alpha(1, \hat{G}_\alpha f)_D = (1 - \alpha R_\alpha 1, f)_D,$$

where \hat{G}_α is the dual resolvent. This implies that $\alpha R_\alpha 1 = 1$ m_D -a.e. for $\alpha > \beta_0$ and consequently q.e. on \bar{D} because $R_\alpha 1$ is quasi-continuous. Equation (6.15) is proven.

(iii) This is an immediate consequence of (i), (ii) as $X^{D,0}$ is the part process of $X^{\bar{D}}$ on D . \square

We conjecture that the property (6.16) for $X^{D,0}$ holds true without the assumption of the relative compactness of D and especially for the minimal process X^0 on E .

Finally, we consider the case where E is \mathbb{R}^d and m is the Lebesgue measure on it. For $\alpha \in (0, 2)$ and an arbitrary open set $D \subset \mathbb{R}^d$, we make use of the Lévy kernel

$$k^{[\alpha]}(x, y) = \frac{\alpha 2^{\alpha-1} \Gamma((\alpha + d)/2)}{\pi^{d/2} \Gamma(1 - \alpha/2)} \frac{1}{|x - y|^{d+\alpha}}, \quad x, y \in \mathbb{R}^d,$$

of the symmetric α -stable process to introduce the Dirichlet form

$$(6.17) \quad \begin{cases} \mathcal{E}_D^{[\alpha]}(u, v) := \iint_{D \times D \setminus \text{diag}} (u(y) - u(x))(v(y) - v(x)) k^{[\alpha]}(x, y) dx dy, \\ \mathcal{F}_D^{[\alpha], r} = \{u \in L^2(D) : u \text{ is Borel measurable and } \mathcal{E}_D^{[\alpha]}(u, u) < \infty\}, \end{cases}$$

on $L^2(D)$ based on the Lebesgue measure on D . Denote by $\mathcal{F}_D^{[\alpha]}$ the $\mathcal{E}_D^{[\alpha]}$ -closure of $C_0^{\text{lip}}(\bar{D})$ in $\mathcal{F}_D^{[\alpha], r}$. For $s \in (0, d]$, a Borel subset Γ of \mathbb{R}^d is said to

be an s -set if there exist positive constants c_1, c_2 such that for all $x \in \Gamma$ and $r \in (0, 1]$, $c_1 r^s \leq \mathcal{H}^s(\Gamma \cap B(x, r)) \leq c_2 r^s$, where \mathcal{H}^s denotes the s -dimensional Hausdorff measure on \mathbb{R}^d and $B(x, r)$ is the ball of radius r centered at $x \in \mathbb{R}^d$.

If the open set D is a d -set, then, by making use of Jonsson–Wallin’s trace theorem [14] as in [7], one can show that $\mathcal{F}_D^{[\alpha]} = \mathcal{F}_D^{[\alpha], r}$ and moreover that a subset of \overline{D} is $\mathcal{E}_D^{[\alpha]}$ -polar iff it is polar with respect to the symmetric α -stable process on \mathbb{R}^d .

Let us consider the kernel $k^{(1)}$ of (1.9) for $w(x)$ given by (5.1) and $\alpha(x)$ satisfying condition (5.2). In particular, it is assumed that

$$0 < \underline{\alpha} \leq \alpha(x) \leq \overline{\alpha} < 2$$

for some constant $\underline{\alpha}, \overline{\alpha}$. $k^{(1)}$ satisfies conditions (2.1)–(2.4) by Proposition 5.1 and one can associate with it the regular lower bounded semi-Dirichlet form η_D (resp., η_D^0) on $L^2(\overline{D}; 1_D dx)$ [resp., $L^2(D)$] possessing as its reference form \mathcal{E}_D (resp., \mathcal{E}_D^0) defined right after (6.1) for $k^{(1)}$ and the Lebesgue measure in place of k and m .

Suppose D is bounded, then there exist positive constants c_3, c_4 with

$$c_3 k^{[\underline{\alpha}]}(x, y) \leq k_s^{(1)}(x, y) \leq c_4 k^{[\overline{\alpha}]}(x, y), \quad x, y \in \overline{D},$$

so that

$$(6.18) \quad c_3 \mathcal{E}_D^{[\underline{\alpha}]}(u, u) \leq \mathcal{E}_D(u, u) \leq c_4 \mathcal{E}_D^{[\overline{\alpha}]}(u, u), \quad u \in C_0^{\text{lip}}(\overline{D}).$$

For the kernel $k^{(1)}$, the Hunt process $X^{\overline{D}}$ on \overline{D} associated with $(\eta_D, \mathcal{F}_{\overline{D}})$ is called a modified reflecting stable-like process, while its part process $X^{D,0}$ on D , which is associated with $(\eta_D^0, \mathcal{F}_D^0)$, is called a censored stable-like process.

PROPOSITION 6.1. *Assume that D is a bounded open d -set.*

(i) *If ∂D is polar with respect to the symmetric $\overline{\alpha}$ -stable process on \mathbb{R}^d , then the censored stable-like process $X^{D,0} = (X_t^{D,0}, P_x, \zeta^0)$ is conservative and it does not approach to ∂D in finite time:*

$$(6.19) \quad P_x(\zeta^0 = \infty) = 1, \quad P_x(X_{t-}^{D,0} \in \partial D \text{ for some } t > 0) = 0.$$

(ii) *If ∂D is nonpolar with respect to the symmetric $\underline{\alpha}$ -stable process on \mathbb{R}^d , then the censored stable-like process $X^{D,0}$ satisfies*

$$(6.20) \quad \int_D P_x(X_{\zeta^0-}^{D,0} \in \partial D, \zeta^0 < \infty) h(x) dx = \int_D P_x(\zeta^0 < \infty) h(x) dx > 0$$

for any strictly positive Borel function h on D with $\int_D h(x) dx = 1$.

PROOF. (i) Since \mathcal{E}_D is a reference form of $(\eta_D, \mathcal{F}_{\overline{D}})$, we see that ∂D is η_D -polar by (6.18) and the stated observation in [7]. The assertions of (i) then follows from Theorem 4.2(ii) and Theorem 6(ii).

(ii) ∂D is not η_D -polar by (6.18) and accordingly not m -polar with respect to the process $X^{\bar{D}}$ by Theorem 4.2(v), where m is the Lebesgue measure on D . Taking Theorem 6.2(i), (iii) into account, we then get (6.20). \square

The polarity of a set $N \subset \mathbb{R}^d$ with respect to the symmetric α -stable process is equivalent to $C^{\alpha/2,2}(N) = 0$ for the Bessel capacity $C^{\alpha/2,2}$ (cf. Section 2.4 of the second edition of [13]). The latter has been well studied in [1] in relation to the Hausdorff measure and the Hausdorff content. For instance, when $\alpha \leq d$ and ∂D is a s -set, ∂D is polar in this sense if and only if $\alpha + s \leq d$. Of course, we get the same results as above for the second kernel $k^{(1)*}$ in (1.9).

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